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The control of bifurcations with engineering applications

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DISSERTATION

**THE CONTROL OF BIFURCATIONS
WITH ENGINEERING APPLICATIONS**

by

Osa E. Fitch
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This dissertation develops a general method for the control of the class of local bifurcations of engineering interest, including saddle-node, transcritical, pitchfork and Hopf bifurcations. The method is based on transforming a general affine single-input control system into quadratic normal form through coordinate transformations and feedback. (The quadratic normal form includes the quadratic order Poincare normal form of the uncontrolled system as a natural subset.) Then, linear and quadratic state feedback control laws are developed which control the shape of the center manifold of the transformed system. It is shown that control of the center manifold allows the quadratic and cubic order terms of the center dynamics to be influenced to produce non-linear stability. Specific matrix operations necessary to transform a general affine single-input control system into quadratic normal form are provided. Specific control laws to stabilize a general system experiencing a linearly unstabilizable saddle-node, transcritical, pitchfork or Hopf bifurcation are also provided.

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THE CONTROL OF BIFURCATIONS WITH ENGINEERING APPLICATIONS

by

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requirements for the degree of

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This dissertation develops a general method for the control of the class of local bifurcations of engineering interest, including saddle-node, transcritical, pitchfork and Hopf bifurcations. The method is based on transforming a general affine single-input control system into quadratic normal form through coordinate transformations and feedback. (The quadratic normal form includes the quadratic order Poincare normal form of the uncontrolled system as a natural subset.) Then, linear and quadratic state feedback control laws are developed which control the shape of the center manifold of the transformed system. It is shown that control of the center manifold allows the quadratic and cubic order terms of the center dynamics to be influenced to produce non-linear stability. Specific matrix operations necessary to transform a general affine single-input control system into quadratic normal form are provided. Specific control laws to stabilize a general system experiencing a linearly unstabilizable saddle-node, transcritical, pitchfork or Hopf bifurcation are also provided.

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It is said that we stand on the shoulders of giants. It is certainly true that this dissertation could not have happened without the efforts of many other people. Knowledge is in books, and no better example of this can be found than the book by Steven H. Strogatz entitled "Nonlinear Dynamics and Chaos, with Applications to Physics, Biology, Chemistry and Engineering", which I wholeheartedly recommend to anyone with an interest in the topic. You will not find a more clearly written introduction to the subject anywhere. On a more personal level, Professor Rick Howard took my cause under his wing as committee chairman in the early stages of a very accelerated Ph.D. program, and went to bat for me time after time, even before I had a clue as to what I was doing. Rick, you got it right, every single time. Professor Wei Kang was everything a student could wish for in a dissertation supervisor, always available, always insightful, and always delightfully cheerful. Wei, you are way ahead of your field, but your ideas are spreading. Dr. Eric Hallberg, my wingman and partner in crime, was always two steps ahead of me in this grueling process. Eric, we made it! My parents, Ken and Myra Fitch, have always supported me in whatever I chose to do, and the last two years have been no exception. Mom and Dad, you saved my bacon more times than you will ever know.

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I. INTRODUCTION AND PRELIMINARY CONCEPTS

A. INTRODUCTION

1. Purpose of this Dissertation

The purpose of this dissertation is to present an organized step-by-step method for the control or stabilization of bifurcations commonly encountered in engineering systems. This dissertation is organized into three main parts. Part I consists of Chapter I and briefly summarizes some preliminary concepts necessary to understand the rest of the material, including simple examples. Part II consists of Chapters II through IV and lays out the process of determining whether a system exhibits a bifurcation, what kind it is and where it occurs, and how to apply multivariable Taylor series expansions and linear control techniques to stabilize the linearly stabilizable part of the system. Part III consists of Chapters V through VIII and introduces material original to this dissertation, specifically a general method for achieving non-linear stabilization of linearly unstabilizable bifurcations. Chapter V defines the concept of a quadratic normal form, a system with simplified quadratic order terms which exhibits dynamics equivalent to the original system, and shows how to transform a system into its quadratic normal form. Chapter VI elaborates on the concept of the center manifold, a reduced dimensional space to which the dynamics of the system collapse, allowing for easier analysis. Chapter VI also shows that systems in the quadratic normal form of Chapter V can control the shape of the center manifold, and Chapter VII shows that this allows the non-linear stabilization of linearly unstabilizable bifurcations. Chapter VII then provides specific state feedback gain formulas which stabilize the commonly encountered types of bifurcations. Chapter VIII works through specific examples which show how to apply the techniques of Chapters V through VII. Taken as a whole, this dissertation lays out a comprehensive method for the control or stabilization of bifurcations commonly encountered in engineering

systems. It should be noted that, throughout this analysis, full state feedback is assumed and the practical question of how to observe the states is not addressed. However, successfully solving the state feedback problem opens the door to further investigation, including output feedback control. Interesting approaches to this problem include that of Gu *et al.* [Ref. 13] and the possibility of integrating the results of this dissertation with existing non-linear filtering theory, as for example, Krener [Ref. 14], Krener [Ref. 15] or Bestle [Ref. 16].

2. Layout of the Chapters

Each chapter is laid out with an introductory section which summarizes the applicable results from previous chapters and introduces the results of the current chapter. The material in the chapter is then developed from that starting point.

3. Original Contribution of this Dissertation

This dissertation introduces some new concepts, and provides some new results. The new results are as follows:

- The general quadratic normal form for single input affine control systems, which is developed in Chapter V, is new. Specific original contributions of Chapter V include: the separation principle; and the individual quadratic normal forms of the controllable/uncontrollable part, the controllable/mixed part, the uncontrollable/mixed part and the uncontrollable/controllable part. Two previous results in this field appear as natural subsets of this theory. The Poincare normal form of a dynamic system without control [Ref. 1] is included as the uncontrollable/uncontrollable part, and Kang's result on the quadratic normal form of a linearly controllable system [Ref. 2] is included as the controllable/controllable part.

- The general method for controlling the shape of the center manifold of a dynamic system, which is developed in Chapter VI, is new. Although the concept of a center manifold is well established in dynamic systems theory (see Wiggins [Ref. 3] or Carr [Ref. 17]), and center manifolds have been used as the basis for control of systems before [Ref. 4], Chapter VI presents a systematic presentation of the general case of control of the shape of the center manifold in the vicinity of an equilibrium point using the quadratic normal form developed in Chapter V.

- The general method for control of a system exhibiting a bifurcation, which is developed in Chapter VII, is new. This method is applicable to all well known local bifurcations in quadratic or cubic order, including saddle-node bifurcations, transcritical and pitchfork bifurcations, Hopf bifurcations, double-zero bifurcations, and cusp bifurcations. Several of these bifurcations have been successfully controlled individually based on the normal form approach [Ref. 2]. In particular, Kang [Ref. 2] controlled bifurcations having at most one linearly uncontrollable mode using a normal form approach. However, this dissertation is the first systematic presentation of how to control a whole class of local bifurcations.
- The general control laws presented for the control of Hopf bifurcations, developed in Chapter VII, are new. Although Abed proved that Hopf bifurcations could be controlled [Ref. 5], general, explicit control laws for Hopf bifurcations do not exist. Others [Ref. 6] have provided specific control laws for specific systems exhibiting Hopf bifurcations, but the results in Chapter VII are the first general control laws for Hopf bifurcations.

4. Who Cares?

The control of bifurcations is of interest to many engineers and scientists in diverse fields. Bifurcations have been observed in aircraft stability and control (wing rock phenomena observed in certain aircraft at high angles of attack), marine engineering (split trajectories of autonomous underwater vehicles), turbine engine compressor design (rotating compressor stall and surge phenomena), and high energy nuclear fusion research (high temperature plasma instabilities). Currently there are no commonly accepted methods for approaching the control of bifurcations in general. This dissertation attempts to fill that gap. Although specific examples will be provided (for example, a simple model of the operation of a turbine engine compressor will be examined in Chapter VIII), the intent of this dissertation is not to solve any specific problem, but rather to present a general approach to the control of bifurcation phenomena which can be applied to many different problems.

B. SOME PRELIMINARY CONCEPTS

This dissertation will be concerned with equations of the form

$$\dot{x} = f(x,\mu) + g(x,\mu)u \tag{I.1}$$

where x is the state vector, μ is the vector of parameters, and $f(x, \mu)$ and $g(x, \mu)$ are general non-linear, vector valued functions of x and μ, and u is a single valued control input. Systems of the form of equation I.1 are known as affine control systems, which means that the control input u can be factored out to stand alone. We will be interested in the case when the control input u has been adjusted so as to trim the system at an equilibrium point. Then, we will examine qualitative changes in the behavior of the system as the vector of parameters μ is varied, which we call bifurcations. (By qualitative change, we mean situations such as a change in the number of equilibrium points, change in the stability of an equilibrium point, creation or destruction of limit cycles, etc.) This dissertation is concerned with developing state feedback control laws for the input u which will either eliminate, stabilize or soften bifurcations which occur as the vector of parameters μ is varied. In order to explain the situation clearly, we will work up to our final equations through informal definitions (explanations) and relatively simple examples. We start our explanation of I.1 by explaining what we mean by state variables and the state vector.

Explanation B.1 (State of a Dynamic System) *We take our definition of state from Ogata [Ref. 7]. The state of a dynamic system is the smallest set of variables (called state variables) such that the knowledge of these variables and the input to the system determines the behavior of the system for all subsequent time. If n state variables are needed to completely describe the behavior of a given system, then these can be considered the n components of a vector x. Such a vector is called a state vector. Note that the state variables are not unique. That is, there may be more than one set of variables which, when taken together, determine the behavior of the system.*

Example. [State Vector] Given the set of coupled dynamic equations

$$\dot{x}_1 = x_1 + x_2 \tag{I.2}$$

$$\dot{x}_2 = -2x_1 + x_2 \tag{I.3}$$

the variables x_1 and x_2 can both be seen to be acceptable state variables since knowledge of both is required to describe the behavior of the system with time. So, the

state vector in this case is

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{I.4}$$

Note that any two independent linear combinations of the variables x_1 and x_2 are also acceptable state variables for this system. ◃

Now let's look at what we mean by a parameter and by a vector of parameters.

Explanation B.2 (Parameter) *A parameter is a value which is constant in any given dynamic system, but which may take on different values from dynamic system to dynamic system. Since a parameter is constant in any given dynamic system, it may be characterized as having a time rate of change of zero (i.e. $\dot{\mu} = 0$). If all the parameters in a given equation or set of equations are stacked up in vector form, the result is called the vector of parameters. If the vector of parameters is appended to the state vector, then the result is known as the appended state vector.*

Example. [Parameters] Given the set of dynamic equations

$$\dot{x}_1 = \mu_1 x_1 \tag{I.5}$$

$$\dot{x}_2 = -2\mu_2 x_2 \tag{I.6}$$

where μ_1 and μ_2 are unspecified constants, it can be seen that either μ_1 or μ_2 can be considered parameters if they can take on different values from dynamic system to dynamic system. In other words, parameters allow us to define a family of dynamic equations, which in this example we can express as

$$\dot{\mu}_1 = 0 \tag{I.7}$$

$$\dot{\mu}_2 = 0 \tag{I.8}$$

$$\dot{x}_1 = \mu_1 x_1 \tag{I.9}$$

$$\dot{x}_2 = -2\mu_2 x_2 \tag{I.10}$$

Here we have μ_1 and μ_2 as parameters, and x_1 and x_2 as state variables, which gives

- The vector of parameters: $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$

- The state vector: $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- The appended state vector: $\begin{bmatrix} \mu \\ x \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ x_1 \\ x_2 \end{bmatrix}$

$\triangleleft$

Now, what do we mean by a vector valued function of x and μ?

Explanation B.3 (Vector Valued Function) *A vector valued function is a vector whose individual components are scalar functions of (possibly) multiple variables, in our case state variables and parameters.*

Example. [Vector Valued Function] Given the set of dynamic equations

$$\dot{x}_1 = \mu_1 x_1 + x_2^2 \tag{I.11}$$

$$\dot{x}_2 = -2\mu_2 x_2 + 3x_1^3 \tag{I.12}$$

we can define two new functions, f_1 and f_2, as

$$f_1(x_1, x_2, \mu_1, \mu_2) = \mu_1 x_1 + x_2^2 \tag{I.13}$$

$$f_2(x_1, x_2, \mu_1, \mu_2) = -2\mu_2 x_2 + 3x_1^3 \tag{I.14}$$

and using the state vector x and the vector of parameters μ, we can re-write the set of dynamic equations as

$$\dot{x}_1 = f_1(x, \mu) \tag{I.15}$$

$$\dot{x}_2 = f_2(x, \mu) \tag{I.16}$$

If we stack the components f_1 and f_2 into a vector, we end up with a vector valued function $f(x, \mu)$ in our set of dynamic equations, i.e.

$$\dot{x} = f(x, \mu) \tag{I.17}$$

where

$$f(x,\mu) = \begin{bmatrix} f_1(x,\mu) \\ f_2(x,\mu) \end{bmatrix} = \begin{bmatrix} \mu_1 x_1 + x_2^2 \\ -2\mu_2 x_2 + 3x_1^3 \end{bmatrix} \tag{I.18}$$

We note that for a fixed value of μ, the vector valued function $f(x,\mu)$ is a vector field. As μ is allowed to vary, $f(x,\mu)$ defines a family of vector fields. ◁

Finally, our definition of single valued control input and its accompanying example will bring us fully up to speed on the equation we started with.

Explanation B.4 (Control Input) *A control input is a variable whose values are fed into a dynamic system from the outside, in order to affect the behavior of that system. The control input is allowed to change with time. A single valued control input has only one value fed in, that is, the control input is a scalar, rather than a vector. Automatic control systems commonly occur in two forms: open loop control, where the control input is predetermined (typically on a time basis), and where the output of the system has no effect on the control input; and feedback control, or closed loop control, where information about the output or state variables of a system is combined according to a pre-set formula and fed in as an input. A third type of control system, operator control or adaptive control, occurs when the system has an intelligent component as the controller (typically a human being, although occasionally applications of robotics or artificial intelligence), who puts in whatever input he chooses, in response to his own situation and perceived need for control. Although in general any of these methods of control may be used, in this dissertation we will only deal with feedback control of a single control input.*

Example. [Affine Control Input] Here we have a dynamic system with a control input, u:

$$\dot{x}_1 = \mu_1 x_1 + x_2^2 + x_2 u \tag{I.19}$$

$$\dot{x}_2 = -2\mu_2 x_2 + 3x_1^3 + u \tag{I.20}$$

If we separate out the terms containing the control input u from the other terms, and stack in vector form, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \mu_1 x_1 + x_2^2 \\ -2\mu_2 x_2 + 3x_1^3 \end{bmatrix} + \begin{bmatrix} x_2 \\ 1 \end{bmatrix} u \tag{I.21}$$

Since the control input u can be factored out by itself, this equation is said to be "affine" in u. Using our vector valued function notation from before, we get

$$\dot{x} = f(x, \mu) + g(x, \mu)u \tag{I.22}$$

where we have

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \tag{I.23}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{I.24}$$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \tag{I.25}$$

$$f(x, \mu) = \begin{bmatrix} \mu_1 x_1 + x_2^2 \\ -2\mu_2 x_2 + 3x_1^3 \end{bmatrix} \tag{I.26}$$

$$g(x, \mu) = \begin{bmatrix} x_2 \\ 1 \end{bmatrix} \tag{I.27}$$

Note that we could have defined the function g as $g(x_2)$ since that is the only variable it depends on. However, it is also permissible to be more general, which we have done here. ◁

Example. [Feedback Control] We decide to apply feedback control to our dynamic system. Assume that we know our state variables, x_1 and x_2, and also our parameters, μ_1 and μ_2. If we desired, we could take combinations of our states and parameters and multiply them by values which we pick (called gains) and feed them back into the control input u. (Note that the combinations can be linear or nonlinear.) Our dynamic system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \mu_1 x_1 + x_2^2 \\ -2\mu_2 x_2 + 3x_1^3 \end{bmatrix} + \begin{bmatrix} x_2 \\ 1 \end{bmatrix} u \tag{I.28}$$

Let's suppose we decide to apply feedback as follows:

$$u = K_1 x_2 + K_2 \mu_2 x_2 + K_3 x_1^3 \tag{I.29}$$

Our dynamic system with feedback applied now looks like

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \mu_1 x_1 + x_2^2 + x_2 \left(K_1 x_2 + K_2 \mu_2 x_2 + K_3 x_1^3\right) \\ K_1 x_2 + (K_2 - 2)\, \mu_2 x_2 + (K_3 + 3)\, x_1^3 \end{bmatrix} \tag{I.30}$$

Let's look at the $\dot{x}_2$ term for a moment. We can pick the gains as follows:

$K_1 = -10$ (This ensures that x_2 will be very exponentially stable close to the equilibrium point.)

$K_2 = 2$ (This cancels the $-2\mu_2 x_2$ term.)

$K_3 = -3$ (This cancels the $3x_1^3$ term.)

Our dynamic system with these feedback gains applied is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \mu_1 x_1 - 9x_2^2 + 2\mu_2 x_2^2 - 3x_1^3 x_2 \\ -10x_2 \end{bmatrix} \tag{I.31}$$

So, in this example we succeeded in stabilizing x_2 with our choice of feedback gains K_1, K_2, and K_3. However, we were left with a non-linear mess for the dynamics of x_1. Is x_1 stable? Is x_1 unstable? How does the stability of x_1 depend on the parameters μ_1 and μ_2? None of the answers to these questions are obvious, because we chose our feedback gains in a very haphazard manner. Determining how to choose our feedback gains to affect the behavior of our system is the point of this dissertation. ◃

This dissertation will present a systematic method for choosing feedback gains (linear and non-linear) which will allow us to stabilize our dynamic system around an equilibrium point, or to tell us when a system cannot be stabilized. In particular, we will focus on the control of bifurcations through the control of the center manifold of a system. Following Kang [Ref. 10], we will extend Poincare's idea of normal forms to systems with control, which will allow us to prove a useful general result about affine non-linear control systems. Putting all of these pieces together will ultimately give us

the desired feedback gains. Since the terms bifurcation, center manifold, and normal form are unfamiliar to many controls engineers, they will be explained in subsequent sections. We start with bifurcations.

C. WHAT IS A BIFURCATION?

What is a bifurcation? Although this dissertation presents a theory for the control of bifurcations, we need to be clear about what a bifurcation is, before we can rush off and try to control one. As in the previous section, we will begin with explanations backed up by examples.

Explanation C.1 (Bifurcation) *We take our definition of bifurcation from Strogatz [Ref. 8]. A bifurcation is a qualitative change in the dynamics of a system as a parameter is varied. The value of the parameter at which the change occurs is called the bifurcation value, also sometimes known as the bifurcation point or point of bifurcation. (In this dissertation all these terms will be used interchangeably.) A bifurcation is always associated with a qualitative change in the nature of the equilibrium points of the system. The qualitative change could be a change in the number of equilibrium points, the change in the stability of an equilibrium point, or other qualitative change, such as the creation or destruction of a limit cycle.*

Explanation C.2 (Equilibrium Points) *An equilibrium point is a value of the state vector x of a dynamic system, such that the time rate of change of the state vector is zero at that point. That is, $\dot{x} = 0$ when $x = x^*$, where x^* is an equilibrium point. For linear systems, the only isolated equilibrium point possible is the origin. However non-linear systems may have more than one isolated equilibrium point, as we will see in the next section. (Non-isolated equilibrium points typically occur in degenerate situations when one or more of the eigenvalues of the linearized system is zero.)*

Example. [Equilibrium Points] Look at the simple dynamic system

$$\dot{x}_1 = \mu_1 x_1 \tag{I.32}$$

where x_1 is the state variable and μ_1 is a parameter. First, let's find all the equilibrium points of this system, which we will call x_1^*. We do this by setting $\dot{x}_1 = 0$, and $x_1 = x_1^*$ and solving the resulting algebraic equation in x_1^*. We get

$$0 = \mu_1 x_1^* \tag{I.33}$$

So, simple-mindedly we announce the answer: all the equilibrium points are given by the equation

$$x_1^* = 0 \tag{I.34}$$

This is indeed true, except for one special case which turns out to be very important. When $\mu = 0$ we get

$$x_1^* = arbitrary \tag{I.35}$$

Clearly, a system with an infinite number of equilibrium points is qualitatively different from a system with only one equilibrium point. Examining the stability of our system we see that the equilibrium point at $x_1 = 0$ is stable for $\mu < 0$ and is unstable for $\mu > 0$, another qualitative change which occurs at $\mu = 0$. We call this qualitative change a bifurcation, and note that $\mu = 0$ is the bifurcation point for this system. (As we will see below, the bifurcation exhibited by this system is a special case of what is known as a transcritical bifurcation.) ◁

Now we will look at specific cases of bifurcations of engineering interest. These include saddle-node bifurcations, transcritical bifurcations, pitchfork bifurcations (supercritical and subcritical), and Hopf bifurcations (supercritical and subcritical). (Figures 1 through 6 were adapted from Strogatz [Ref. 8].)

1. Saddle-Node Bifurcation

Example. [Saddle-Node Bifurcation] Look at a simple non-linear dynamic system which exhibits a saddle-node bifurcation

$$\dot{x}_1 = \mu_1 + x_1^2 \tag{I.36}$$

where x_1 is the dynamic state and μ_1 is a parameter. Again we find the equilibrium points as $x = x^*$ such that $\dot{x} = 0$, i.e.

$$0 = \mu_1 + x_1^{*2} \tag{I.37}$$

which gives

$$x_1^* = \pm\sqrt{-\mu_1} \tag{I.38}$$

an answer which only exists for $\mu_1 \leq 0$. Note that no equilibrium points exist for $\mu_1 > 0$. A qualitative change in the dynamics of the system occurs at $\mu_1 = 0$, and we call this change a saddle-node bifurcation. If we were to examine the stability of the two equilibrium points (when they exist) we would find that the equilibrium point at $x_1^* = -\sqrt{-\mu_1}$ was stable, while the equilibrium point at $x_1^* = \sqrt{-\mu_1}$ was unstable. In this system, as μ_1 is increased and passes through $\mu_1 = 0$, the stable and unstable equilibrium points coalesce and annihilate each other. The coalescence and annihilation of a pair of equilibrium points (or conversely the creation and separation of a pair of equilibrium points) — one stable and one unstable — is the first basic bifurcation mechanism in non-linear systems. We can represent the behavior of the equilibrium points on a bifurcation diagram, as shown in Figure 1. Here, the horizontal axis is

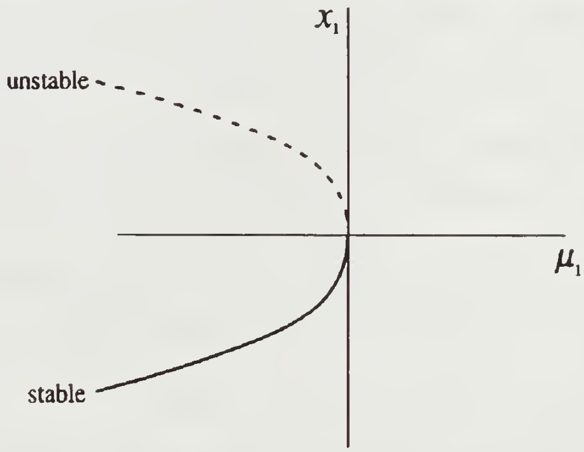

Figure 1. Bifurcation Diagram for a Saddle-Node Bifurcation

the value of the parameter μ_1, and the vertical axis is the location of any equilibrium points x_1^*, if they exist. The stability of the equilibrium points is also shown on the diagram, with the stable equilibrium point indicated by the solid branch of the curve, and the unstable equilibrium point indicated by the dashed branch of the curve. The

bifurcation is clearly visible at the origin, where the two branches of the curve meet and annihilate, resulting in no equilibrium points for $\mu_1 > 0$. ◁

2. Transcritical Bifurcation

Example. [Transcritical Bifurcation] Now look at simple non-linear dynamic system which exhibits a transcritical bifurcation. Our system is

$$\dot{x}_1 = \mu_1 x_1 + x_1^2 \tag{I.39}$$

where x_1 is the state variable and μ_1 is a parameter. Again we find the equilibrium points as $x = x^*$ such that $\dot{x} = 0$, i.e.

$$0 = \mu_1 x_1^* + x_1^{*2} \tag{I.40}$$

which gives

$$x_1^* = 0 \tag{I.41}$$

$$x_1^* = -\mu_1 \tag{I.42}$$

Here we see that there are two equilibrium points, except when $\mu_1 = 0$ and there is only one, a qualitative change in the system. Examining the stability of the equilibrium point at $x_1^* = 0$ (by using the first method of Lyapunov and examining the linearized system) we see that $x_1^* = 0$ is a stable equilibrium point for $\mu_1 < 0$ and an unstable equilibrium point for $\mu_1 > 0$, another qualitative change. (If we were to examine the stability of the equilibrium point at $x_1^* = -\mu_1$ we would find that it also switched stability at $\mu_1 = 0$, and that for any given value of μ_1 its stability was opposite to that of the other equilibrium point.) This is called a transcritical bifurcation. We can again represent the behavior of the equilibrium points on a bifurcation diagram, as shown in Figure 2. Again, the horizontal axis is the value of the parameter μ_1, and the vertical axis is the location of the equilibrium points x_1^*. The stable equilibrium point is again indicated by the solid branch, and the unstable equilibrium point indicated by the dashed branch. The bifurcation is clearly visible

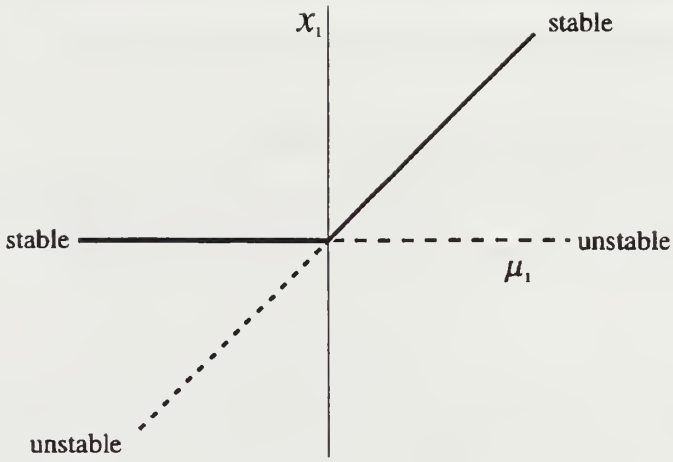

Figure 2. Bifurcation Diagram for a Transcritical Bifurcation

at the origin, where the two branches of the curve meet and the stability of each branch switches. ◁

3. Pitchfork Bifurcations

Example. [Supercritical Pitchfork Bifurcation] Now look at a simple non-linear dynamic system which exhibits a supercritical pitchfork bifurcation. Our system is

$$\dot{x}_1 = \mu_1 x_1 - x_1^3 \tag{I.43}$$

where x_1 is the dynamic state and μ_1 is a parameter. Again we find the equilibrium points as $x = x^*$ such that $\dot{x} = 0$, i.e.

$$0 = \mu_1 x_1^* - x_1^{*3} \tag{I.44}$$

which gives

$$x_1^* = 0 \tag{I.45}$$

$$x_1^* = \pm\sqrt{\mu_1} \tag{I.46}$$

Here we see that the equilibrium point at $x_1^* = 0$ always exists regardless of the value of μ, but that the equilibrium points at $x_1^* = \pm\sqrt{\mu_1}$ only exist for $\mu_1 > 0$. Thus, a

qualitative change in the dynamics of this system occurs at $\mu_1 = 0$, which we refer to as a supercritical pitchfork bifurcation. (The next example will illustrate a subcritical pitchfork bifurcation.) To determine the stability of our system by using the first method of Lyapunov, we examine the Jacobian matrix of our system evaluated at each equilibrium point. The Jacobian is given by

$$J = \frac{\partial}{\partial x_1}\left(\mu_1 x_1 - x_1^3\right)_{x_1 = x_1^*} = \mu_1 - 3x_1^{*2} \tag{I.47}$$

so we have

$$J = \mu_1 \tag{I.48}$$

for $x_1^* = 0$, and

$$J = -2\mu_1 \tag{I.49}$$

for $x_1^* = \pm\sqrt{\mu_1}$. Since our system is one-dimensional, the stability (sign of the eigenvalue) is immediate by inspection: our equilibrium point at $x_1^* = 0$ is stable for $\mu_1 < 0$ and unstable for $\mu_1 > 0$; and our equilibrium points at $x_1^* = \pm\sqrt{\mu_1}$ are stable for $\mu_1 > 0$. (Note that since they do not exist, we can not evaluate their stability for $\mu_1 < 0$.) We can again represent the behavior of the equilibrium points on a bifurcation diagram, as shown in Figure 3. Again, the horizontal axis is the value of the parameter μ_1, and the vertical axis is the location of the equilibrium points x_1^*. The stable equilibrium points are again indicated by the solid branches, and the unstable equilibrium point is indicated by the dashed branch. The bifurcation is clearly visible at the origin, where the two stable branches of the curve split off and the stability of the origin switches from stable to unstable. So, to summarize, for $\mu_1 < 0$, we have one equilibrium point at $x_1 = 0$, which is stable. This situation persists, with no qualitative changes to our dynamics, until $\mu_1 > 0$, when our equilibrium point at $x_1 = 0$ becomes unstable, which is a qualitative change. But because two new stable equilibrium points at $x_1 = \pm\sqrt{\mu_1}$ simultaneously come into existence on either side of the unstable equilibrium point, the dynamics around $x_1 = 0$ are bounded. In this situation the bifurcation is said to be supercritical, or "soft". ◁

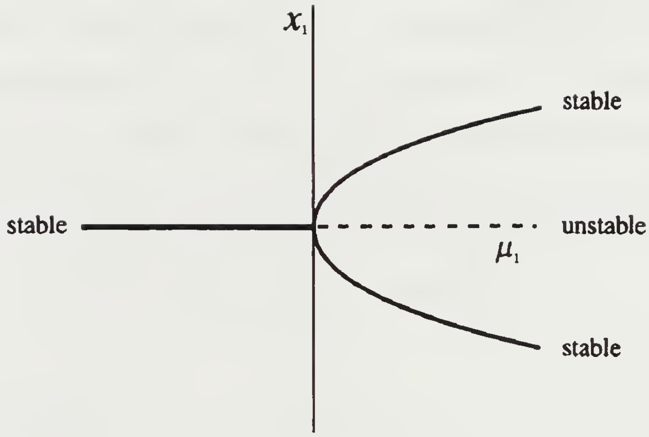

Figure 3. Bifurcation Diagram for a Supercritical Pitchfork Bifurcation

Example. [Subcritical Pitchfork Bifurcation] Look at another simple non-linear dynamic system

$$\dot{x}_1 = \mu_1 x_1 + x_1^3 \tag{I.50}$$

where x_1 is the dynamic state and μ_1 is a parameter. Again we find the equilibrium points as $x = x^*$ such that $\dot{x} = 0$, i.e.

$$0 = \mu_1 x_1^* + x_1^{*3} \tag{I.51}$$

which gives

$$x_1^* = 0 \tag{I.52}$$

$$x_1^* = \pm\sqrt{-\mu_1} \tag{I.53}$$

Here we see that the equilibrium point at $x_1^* = 0$ always exists regardless of the value of μ_1, but that the equilibrium points at $x_1^* = \pm\sqrt{-\mu_1}$ only exist for $\mu_1 < 0$. Thus, a qualitative change in the dynamics of this system occurs at $\mu_1 = 0$, which we refer to as a subcritical pitchfork bifurcation. To determine the stability of our system by using the first method of Lyapunov, we examine the Jacobian matrix of our system

evaluated at each equilibrium point. The Jacobian is given by

$$J = \frac{\partial}{\partial x_1}\left(\mu_1 x_1 + x_1^3\right)_{x_1 = x_1^*} = \mu_1 + 3x_1^{*2} \tag{I.54}$$

so we have

$$J = \mu_1 \tag{I.55}$$

for $x_1^* = 0$, and

$$J = -2\mu_1 \tag{I.56}$$

for $x_1^* = \pm\sqrt{-\mu_1}$. Since our system is one-dimensional, the stability (sign of the eigenvalue) is immediate by inspection: our equilibrium point at $x_1^* = 0$ is stable for $\mu_1 < 0$ and unstable for $\mu_1 > 0$; and our equilibrium points at $x_1^* = \pm\sqrt{-\mu_1}$ are unstable for $\mu_1 < 0$. (Note that since they do not exist, we can not evaluate their stability for $\mu_1 > 0$.) We can again represent the behavior of the equilibrium points on a bifurcation diagram, as shown in Figure 4. Again, the horizontal axis is

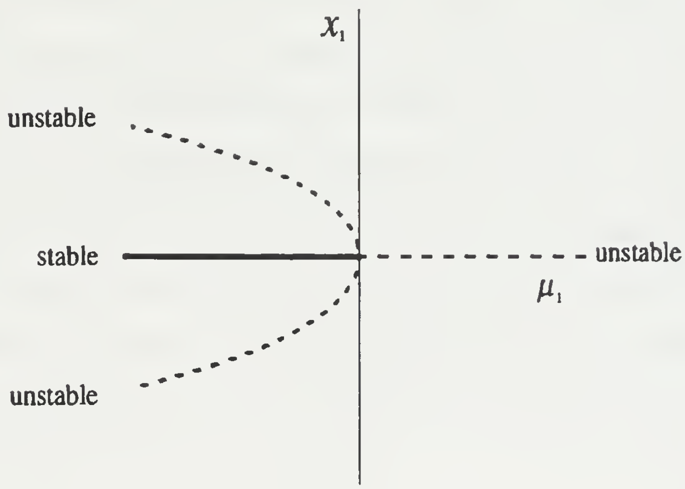

Figure 4. Bifurcation Diagram for a Subcritical Pitchfork Bifurcation

the value of the parameter μ_1, and the vertical axis is the location of the equilibrium points x_1^*. The stable equilibrium point is again indicated by the solid branch, and the unstable equilibrium points are indicated by the dashed branches. The bifurcation is

clearly visible at the origin, where the two unstable branches of the curve meet and the stability of the origin switches from stable to unstable. So, to summarize, for $\mu_1 < 0$, we have one equilibrium point at $x_1 = 0$, which is stable, and two additional equilibrium points at $x_1 = \pm\sqrt{-\mu_1}$, which are unstable. This situation persists, with no qualitative changes to our dynamics, until $\mu_1 > 0$, when our equilibrium point at $x_1 = 0$ becomes unstable, which is a qualitative change, and the other two equilibrium points cease to exist, which is another qualitative change. Now, the dynamics around $x_1 = 0$ are unbounded, and trajectories around the equilibrium point diverge exponentially, with nothing to "catch" them. In this situation the bifurcation is said to be subcritical, or "hard". $\triangleleft$

4. Hopf Bifurcations

Example. [Supercritical Hopf Bifurcation] Now look at a non-linear dynamic system which exhibits a supercritical Hopf bifurcation. Our system in Cartesian coordinates is

$$\dot{x}_1 = \mu_1 x_1 - x_2 - x_1\left(x_1^2 + x_2^2\right) \quad \text{(I.57)}$$

$$\dot{x}_2 = x_1 + \mu_1 x_2 - x_2\left(x_1^2 + x_2^2\right) \quad \text{(I.58)}$$

and in polar coordinates is

$$\dot{r} = \mu_1 r - r^3 \quad \text{(I.59)}$$

$$\dot{\theta} = 1 \quad \text{(I.60)}$$

with

$$x_1 = r\cos\theta \quad \text{(I.61)}$$

$$x_2 = r\sin\theta \quad \text{(I.62)}$$

where x_1 and x_2 (or r and θ) are state variables and μ_1 is a parameter. Again we find the equilibrium points by solving for $x = x^*$ such that $\dot{x} = 0$, i.e.

$$0 = \mu_1 x_1^* - x_2^* - x_1^*\left(x_1^{*2} + x_2^{*2}\right) \quad \text{(I.63)}$$

$$0 = x_1^* + \mu_1 x_2^* - x_2^* \left(x_1^{*2} + x_2^{*2} \right) \tag{I.64}$$

which gives an immediate answer

$$x_1^* = 0 \tag{I.65}$$

$$x_2^* = 0 \tag{I.66}$$

However, although we have found an equilibrium point, how do we know there aren't more? For general systems, this is not an easy task, as we will see in Chapter II. For this system however, we can look at the equation in polar coordinates, where we see that there is only a single equilibrium point, at $r^* = 0$, which corresponds to the previously determined equilibrium point $x_1^* = x_2^* = 0$. (Note that technically $r^* = 0$ is not an equilibrium point, since $\dot{\theta} \neq 0$. However, this is just an artifact of our coordinate system. But, it does allow us to rule out the existence of any other equilibrium points which are not at the origin.) So, we have only one equilibrium point, and its existence does not depend on the value of the parameter μ_1. So there are no qualitative changes due to a change in the number of equilibrium points. Does the stability of the equilibrium point depend on the value of μ_1? Looking at the system in polar coordinates, it is immediately obvious that it does (the r dynamics undergoes a supercritical pitchfork bifurcation). However, can we see this by examining our original system? We can determine the stability of our system by using the first method of Lyapunov, and examine the Jacobian matrix of our system evaluated at the equilibrium point. The Jacobian is given by

$$J = \left(\frac{\partial f}{\partial x} \right)_{x=x^*} \tag{I.67}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{I.68}$$

and

$$f(x) = \begin{bmatrix} \mu_1 x_1 - x_2 - x_1 \left(x_1^2 + x_2^2 \right) \\ x_1 + \mu_1 x_2 - x_2 \left(x_1^2 + x_2^2 \right) \end{bmatrix} \tag{I.69}$$

so we get

$$J = \begin{bmatrix} \mu_1 - 3x_1^2 - x_2^2 & -1 - 2x_1x_2 \\ 1 - 2x_1x_2 & \mu_1 - x_1^2 - 3x_2^2 \end{bmatrix}_{x=0} = \begin{bmatrix} \mu_1 & -1 \\ 1 & \mu_1 \end{bmatrix} \tag{I.70}$$

When we calculate the eigenvalues λ of our Jacobian matrix J with the formula

$$\det(\lambda I - J) = 0 \tag{I.71}$$

we get

$$\lambda = \mu_1 \pm i \tag{I.72}$$

So, we can see that the stability of our equilibrium point at the origin (which is determined by the sign of the real part of the eigenvalues) changes at $\mu_1 = 0$, and indeed that the origin is exponentially stable for $\mu_1 < 0$, and unstable for $\mu_1 > 0$. Now, unlike the pitchfork bifurcations above, the stability of this equilibrium point did not change due to the coalescence of two other equilibrium points — there are no other equilibrium points in the system. So what did happen? Looking at equations I.59 we can see that the r dynamic equation undergoes a supercritical pitchfork bifurcation when $\mu_1 = 0$, and a new, stable r–equilibrium is established at $r^* = \sqrt{\mu_1}$ for $\mu_1 > 0$. (Note that we do not speak of an additional r–equilibrium at $r^* = -\sqrt{\mu_1}$, since negative radius is meaningless.) But since the θ dynamic equation only allows for an equilibrium point at the origin, our new, stable r–equilibrium is not a stationary point, it is point moving on a fixed trajectory — a limit cycle. It is the creation of this stable limit cycle which is associated with the change of stability of the equilibrium point at the origin, and because the stable limit cycle surrounds the unstable equilibrium point, the dynamics around the origin are bounded, as shown in Figure 5. In this situation the bifurcation is said to be supercritical, or "soft". In this system, at $\mu_1 = 0$, the creation of a stable limit cycle around an equilibrium point caused the stability of the equilibrium point at the origin to switch from stable to unstable. The change of stability of an equilibrium point associated with the creation or destruction of a limit cycle is the second basic bifurcation mechanisms in non-linear systems. ◁

Figure 5. Supercritical Hopf Bifurcation

Example. [Subcritical Hopf Bifurcation] Now look at a non-linear dynamic system which exhibits a subcritical Hopf bifurcation. Our system in Cartesian coordinates is

$$\dot{x}_1 = \mu_1 x_1 - x_2 + x_1 \left(x_1^2 + x_2^2\right) \tag{I.73}$$

$$\dot{x}_2 = x_1 + \mu_1 x_2 + x_2 \left(x_1^2 + x_2^2\right) \tag{I.74}$$

and in polar coordinates is

$$\dot{r} = \mu_1 r + r^3 \tag{I.75}$$

$$\dot{\theta} = 1 \tag{I.76}$$

with

$$x_1 = r\cos\theta \tag{I.77}$$

$$x_2 = r\sin\theta \tag{I.78}$$

where x_1 and x_2 (or r and θ) are state variables and μ_1 is a parameter. Again we find the equilibrium points by solving for $x = x^*$ such that $\dot{x} = 0$, i.e.

$$0 = \mu_1 x_1^* - x_2^* + x_1^* \left(x_1^{*2} + x_2^{*2}\right) \tag{I.79}$$

$$0 = x_1^* + \mu_1 x_2^* + x_2^* \left(x_1^{*2} + x_2^{*2}\right) \tag{I.80}$$

which gives the same answer as the supercritical Hopf bifurcation

$$x_1^* = 0 \tag{I.81}$$

$$x_2^* = 0 \tag{I.82}$$

Again, as before, the equation in polar coordinates confirms that the origin is the only equilibrium point. Now we check to see if the stability of the origin depends on the value of the parameter μ_1. Again the equation in polar coordinates confirms that it does, since the r dynamics undergoes a subcritical pitchfork bifurcation. We also determine the stability of our original system by using the first method of Lyapunov and examining the Jacobian matrix of our system, evaluated at the origin. The Jacobian is given by

$$J = \left(\frac{\partial f}{\partial x}\right)_{x=x^*} \tag{I.83}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{I.84}$$

and

$$f(x) = \begin{bmatrix} \mu_1 x_1 - x_2 + x_1\left(x_1^2 + x_2^2\right) \\ x_1 + \mu_1 x_2 + x_2\left(x_1^2 + x_2^2\right) \end{bmatrix} \tag{I.85}$$

so we get

$$J = \begin{bmatrix} \mu_1 + 3x_1^2 + x_2^2 & -1 + 2x_1x_2 \\ 1 + 2x_1x_2 & \mu_1 + x_1^2 + 3x_2^2 \end{bmatrix}_{x=0} = \begin{bmatrix} \mu_1 & -1 \\ 1 & \mu_1 \end{bmatrix} \tag{I.86}$$

which is exactly the same result we got for the supercritical Hopf bifurcation. So, the eigenvalues are

$$\lambda = \mu_1 \pm i \tag{I.87}$$

which give us the same stability criteria as before, that is, the origin is exponentially stable for $\mu_1 < 0$, and unstable for $\mu_1 > 0$. Now, so far, this example would seem to be a waste — nothing new has been discovered. But, when we look at equation I.75 we can see that the r dynamic equation undergoes a subcritical pitchfork bifurcation

when $\mu_1 = 0$. For $\mu_1 < 0$ an unstable r–equilibrium exists at $r^* = \sqrt{-\mu_1}$. (Note that again we do not speak of an additional r–equilibrium at $r^* = -\sqrt{-\mu_1}$, since negative radius is meaningless.) And again, since the θ dynamic equation only allows for an equilibrium point at the origin, our unstable r–equilibrium is not a point, it is a limit cycle. It is an unstable limit cycle surrounding a stable equilibrium point, and it is the collapse of this unstable limit cycle onto the stable equilibrium point which is associated with the change of stability of the equilibrium point at the origin, as shown in Figure 6. Also, because the unstable limit cycle is annihilated at the

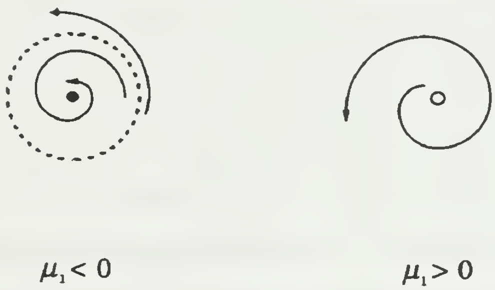

Figure 6. Subcritical Hopf Bifurcation

point of bifurcation (when the stability of the origin changes), the dynamics around the origin become unbounded, and trajectories around the equilibrium point diverge exponentially, with nothing to "catch" them. In this situation the bifurcation is said to be subcritical, or "hard". Much of the rest of this dissertation will be concerned with ways to "soften" bifurcations, that is to use control inputs to turn subcritical bifurcations into a supercritical bifurcations. ◁

D. WHAT IS A CENTER MANIFOLD?

What is a center manifold? What is an invariant manifold? To most engineers, a manifold is a pipe. It may be straight, or it may be curved, but regardless of how the pipe twists or curves, what is in the pipe stays in the pipe, from one end to the

other — it doesn't come squirting out the side somewhere in between. Now, we are dealing with mathematics, not hardware in this dissertation (you won't find a piece of real pipe mentioned anywhere, not even in the appendices) but we can make a strong analogy which justifies our use of the terms: In dynamic systems, the manifold (piece of pipe) we are concerned with is a surface inside the state space, which has the property that trajectories which start on the surface, stay on the surface, just as our real flow stays in the pipe. This property (staying on the surface, or manifold) is called invariance, and in any given space there are as many invariant manifolds as there are trajectories. By itself then, the definition of an invariant manifold is not very useful. But we are not interested in just any invariant manifold — we are only interested in one particular invariant manifold, the center manifold. So what is the center manifold? The center manifold is that invariant manifold which is the "best match" to the center subspace of an equilibrium point. What is the center subspace? The center subspace is defined for the linearization around an equilibrium point as being that subspace spanned by the generalized eigenvectors having eigenvalues with zero real parts. Again, we use informal definitions and examples to make our point.

Explanation D.1 (Invariant Manifold) *An invariant manifold is a surface (hypersurface) inside a state space which has the property that trajectories (a trajectory is a path traced out by a point over time) on the manifold remain on the manifold.*

Explanation D.2 (Center Manifold and Center Subspace) *The center manifold is that invariant manifold which has the same dimension as the center subspace of an equilibrium point, and is tangent to the center subspace at that equilibrium point. The center subspace is that subspace spanned by the generalized eigenvectors of the linearization around the equilibrium point which have eigenvalues with zero real parts.*

Example. [Center Subspace] Let's look at an extremely simple linear system. The system

$$\dot{x}_1 = -x_1 \tag{I.88}$$

$$\dot{x}_2 = 0 \tag{I.89}$$

can be expressed in vector/matrix form as

$$\dot{x} = Jx \tag{I.90}$$

with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{I.91}$$

and

$$J = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \tag{I.92}$$

where we have used J to stand for the Jacobian matrix of the system. The eigenvalues of J are $\lambda_1 = -1$ and $\lambda_2 = 0$, which gives the generalized eigenvectors as

$$V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{I.93}$$

$$V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{I.94}$$

where we have normalized both generalized eigenvectors. Now, since the center subspace is the subspace spanned by the generalized eigenvectors corresponding to the eigenvalues with zero real parts, our center subspace for this example is spanned by V_2, and is defined by the equation

$$x_1 = 0 \tag{I.95}$$

◁

Example. [Center Manifold] Now look at the non-linear system

$$\dot{x}_1 = -x_1 + x_2^2 \tag{I.96}$$

$$\dot{x}_2 = x_1^2 \tag{I.97}$$

The linearization of this system around the origin is

$$\dot{x}_1 = -x_1 + O^{(2)} \tag{I.98}$$

$$\dot{x}_2 = 0 + O^{(2)} \tag{I.99}$$

which is exactly the same as the linear system in the previous example. So, we would like to calculate the center manifold of our non-linear system, to quadratic order. How would we do it? We proceed following the method of Carr [Ref. 17] by calculating the dynamics of trajectories on the center manifold two ways and equating them. The surface where the two answers are equal is our center manifold. (Note that since our space is two-dimensional, and since the center subspace is one-dimensional, our center manifold "surface" is going to be a one-dimensional curve.) First, we calculate the dynamics defined by the gradient of the center manifold. Then, we calculate the dynamics in general, and restrict them to the center manifold. Since the x_2 axis spans the center subspace, we take x_2 as our independent variable, and calculate the surface

$$x_{1_{cm}} = \Omega(x_2) \tag{I.100}$$

as our center manifold. Since we would like to calculate this surface to quadratic order, we expand in a Taylor series to get

$$x_{1_{cm}} = \Omega_L x_2 + \Omega_Q x_2^2 + O^{(3+)} \tag{I.101}$$

where Ω_L and Ω_Q are the linear and quadratic coefficients of $\Omega(x_2)$, respectively. Now, calculating the x_1 dynamics as defined by the gradient of the center manifold, we get

$$\begin{aligned} \dot{\Omega} &= \frac{\partial \Omega}{\partial x_2}\dot{x}_2 = \left(\Omega_L + 2\Omega_Q x_2 + O^{(2+)}\right)\dot{x}_2 \\ &= \left(\Omega_L + 2\Omega_Q x_2 + O^{(2+)}\right)x_1^2 \\ &= \left(\Omega_L + 2\Omega_Q x_2 + O^{(2+)}\right)\left(\Omega_L x_2 + \Omega_Q x_2^2 + O^{(3+)}\right)^2 \end{aligned} \tag{I.102}$$

or

$$\dot{\Omega} = \Omega_L^3 x_2^2 + O^{(3+)} \tag{I.103}$$

Calculating the x_1 dynamics in general, and then restricting them to the center manifold gives

$$\begin{aligned} \dot{x}_{1_{cm}} &= -x_{1_{cm}} + x_2^2 \\ &= -\left(\Omega_L x_2 + \Omega_Q x_2^2 + O^{(3+)}\right) + x_2^2 \end{aligned} \tag{I.104}$$

or

$$\dot{x}_{1_{cm}} = -\Omega_L x_2 + (1 - \Omega_Q)\, x_2^2 + O^{(3+)} \tag{I.105}$$

Now, if we equate the linear and quadratic terms in equations I.103 and I.105, we can calculate the coefficients Ω_L and Ω_Q as

$$\Omega_L = 0 \tag{I.106}$$

$$\Omega_Q = 1 \tag{I.107}$$

which gives the equation for our center manifold surface to quadratic order as

$$x_{1_{cm}} = x_2^2 + O^{(3+)} \tag{I.108}$$

We can see that this is tangent to the center subspace of our linearization

$$x_{1_{cs}} = 0 \tag{I.109}$$

by the fact that the linear terms of our center manifold surface are zero. ◃

E. WHAT IS A NORMAL FORM?

The theory of normal forms was initiated by the famous mathematician Henri Poincare over 100 years ago [Ref. 9]. It has been expanded and updated over the years, including work by Takens [Ref. 18]. More recently, Kang [Ref. 10] has applied the theory to systems having control. Briefly stated, the theory of normal forms reveals how much of the non-linearity of a system is inherent in the system, and how much of the non-linearity can be removed by appropriate coordinate transformations and (in the case of systems with control) by feedback. Quoting Wiggins [Ref. 1]

> The method of normal forms provides a way of finding a coordinate system in which the dynamical system takes the "simplest" form, where the term "simplest" will be defined as we go along. As we develop the method, three important characteristics should become apparent.
>
> 1. The method is local in the sense that the coordinate transformations are generated in a neighborhood of a known solution. For our purposes, the known solution will be an equilibrium point.

2. In general, the coordinate transformations will be non-linear functions of the dependent variables. However, the important point is that these coordinate transformations are found by solving a sequence of linear problems.

3. The structure of the normal form is determined entirely by the nature of the linear part of the vector field.

Again, definitions and examples follow.

Explanation E.1 (Normal Form) *The normal form of a dynamic system is the residue remaining after all possible terms have been cancelled through the use of coordinate transformations and feedback (if the system includes control). Due to the different possible ways to cancel terms, the normal form of a system is not unique.*

Example. [Normal Form without Control] Look at the simple system

$$\dot{x}_1 = -x_1 + 2x_1^2 \tag{I.110}$$

Suppose that we wanted to perform a coordinate transformation to try and eliminate the quadratic term $2x_1^2$. How would we go about it? Suppose, for the moment, that we happened to be really good at guessing the answer, and we guessed that a coordinate transformation of the form

$$x_1 = \tilde{x}_1 - 2\tilde{x}_1^2 \tag{I.111}$$

would work. (We will see in the next example where this transformation came from.) Now, let's calculate $\dot{x}_1$ two ways and set them equal to one another. The first way is to plug our coordinate transformation (I.111) into our original system (I.110), which gives

$$\begin{aligned} \dot{x}_1 &= -\left(\tilde{x}_1 - 2\tilde{x}_1^2\right) + 2\left(\tilde{x}_1 - 2\tilde{x}_1^2\right)^2 \\ &= -\tilde{x}_1 + 4\tilde{x}_1^2 - 8\tilde{x}_1^3 + 8\tilde{x}_1^4 \end{aligned} \tag{I.112}$$

The second way is to differentiate our coordinate transformation itself, which gives

$$\begin{aligned} \dot{x}_1 &= \dot{\tilde{x}}_1 - 4\tilde{x}_1\dot{\tilde{x}}_1 \\ &= (1 - 4\tilde{x}_1)\,\dot{\tilde{x}}_1 \end{aligned} \tag{I.113}$$

Now, setting these two equal to one another, and rearranging gives

$$\dot{\tilde{x}}_1 = \frac{-\tilde{x}_1 + 4\tilde{x}_1^2 - 8\tilde{x}_1^3 + 8\tilde{x}_1^4}{1 - 4\tilde{x}_1} \tag{I.114}$$

which on first glance is no improvement at all. But that's because we haven't dealt with the term $(1 - 4\tilde{x}_1)^{-1}$ yet. If we series expand $(1 - 4\tilde{x}_1)^{-1}$ around the equilibrium point of interest, $\tilde{x}_1 = 0$, we get, to first order

$$(1 - 4\tilde{x}_1)^{-1} = 1 + 4\tilde{x}_1 + O^{(2+)} \tag{I.115}$$

which we can plug in to get

$$\begin{aligned} \dot{\tilde{x}}_1 &= \left(1 + 4\tilde{x}_1 + O^{(2+)}\right)\left(-\tilde{x}_1 + 4\tilde{x}_1^2 - 8\tilde{x}_1^3 + 8\tilde{x}_1^4\right) \\ &= -\tilde{x}_1 + 4\tilde{x}_1^2 - 4\tilde{x}_1^2 + O^{(3+)} \\ &= -\tilde{x}_1 + O^{(3+)} \end{aligned} \tag{I.116}$$

So, we have eliminated the quadratic term, at the expense of added complexity in the higher order terms. The normal form for our original system (I.110), to quadratic order, is

$$\dot{\tilde{x}}_1 = -\tilde{x}_1 + O^{(3+)} \tag{I.117}$$

◁

Now, this example demonstrated what we mean by the normal form, but left two large questions unanswered: How did we come up with the coordinate transformation we used; and how do we deal with the inverse operation which seems to be an integral part of the process? The next example will illustrate how we handle these two problems.

Example. [Normal Form without Control, Reprise] Look again at the simple system from the previous example

$$\dot{x}_1 = -x_1 + 2x_1^2 \tag{I.118}$$

Again, suppose that we wanted to perform a coordinate transformation to try and eliminate the quadratic term $2x_1^2$. Without knowing the answer ahead of time, how would we go about it? One way is to use a coordinate transformation which includes all possible quadratic terms with unknown but constant coefficients, and then try and pick the coefficients at the end so that the desired cancellation occurs. In this one dimensional example, the only possible quadratic term is $\tilde{x}_1^2$, so we use a coordinate transformation of the form

$$x_1 = \tilde{x}_1 + h_1\tilde{x}_1^2 \tag{I.119}$$

where h_1 is an unknown constant coefficient which we will determine later. Proceeding as in the previous example we have

$$\begin{aligned} \dot{x}_1 &= -\left(\tilde{x}_1 + h_1\tilde{x}_1^2\right) + 2\left(\tilde{x}_1 + h_1\tilde{x}_1^2\right)^2 \\ &= -\tilde{x}_1 + (2 - h_1)\,\tilde{x}_1^2 + 4h_1\tilde{x}_1^3 + 2h_1^2\tilde{x}_1^4 \end{aligned} \tag{I.120}$$

and

$$\begin{aligned} \dot{x}_1 &= \dot{\tilde{x}}_1 + 2h_1\tilde{x}_1\dot{\tilde{x}}_1 \\ &= (1 + 2h_1\tilde{x}_1)\,\dot{\tilde{x}}_1 \end{aligned} \tag{I.121}$$

which gives

$$\dot{\tilde{x}}_1 = (1 + 2h_1\tilde{x}_1)^{-1}\left(-\tilde{x}_1 + (2 - h_1)\,\tilde{x}_1^2 + 4h_1\tilde{x}_1^3 + 2h_1^2\tilde{x}_1^4\right) \tag{I.122}$$

Now we have to deal with how to find the quantity $(1 + 2h_1\tilde{x}_1)^{-1}$ in general. One way is to let

$$\phi(\tilde{x}_1) = (1 + 2h_1\tilde{x}_1)^{-1} \tag{I.123}$$

and expand $\phi(\tilde{x}_1)$ in a Taylor series, which we represent as

$$\phi(\tilde{x}_1) = \phi_0 + \phi_1\tilde{x}_1 + \phi_2\tilde{x}_1^2 + \ldots \tag{I.124}$$

Then, since a quantity times its inverse is one (or the identity matrix for higher dimensional cases) we get

$$(1 + 2h_1\tilde{x}_1)\,\phi(\tilde{x}_1) = (1 + 2h_1\tilde{x}_1)\left(\phi_0 + \phi_1\tilde{x}_1 + \phi_2\tilde{x}_1^2 + \ldots\right) \tag{I.125}$$

$$
\begin{aligned}
&= (\phi_0) + (2h_1\phi_0 + \phi_1)\,\tilde{x}_1 + (2h_1\phi_1 + \phi_2)\,\tilde{x}_1^2 + \ldots \\
&= 1
\end{aligned}
$$

So, we solve for ϕ_0, ϕ_1, ϕ_2, etc. by matching coefficients term by term. We get

$$\phi_0 = 1 \tag{I.126}$$

$$(2h_1\phi_0 + \phi_1) = 0 \tag{I.127}$$

$$(2h_1\phi_1 + \phi_2) = 0 \tag{I.128}$$

$$\vdots$$

which gives

$$\phi_0 = 1 \tag{I.129}$$

$$\phi_1 = -2h_1 \tag{I.130}$$

$$\phi_2 = 4h_1^2 \tag{I.131}$$

$$\vdots$$

yielding

$$(1 + 2h_1\tilde{x}_1)^{-1} = 1 - 2h_1\tilde{x}_1 + 4h_1^2\tilde{x}_1^2 + O^{(3+)} \tag{I.132}$$

When we plug this into equation I.122 we get

$$
\begin{aligned}
\dot{\tilde{x}}_1 &= \left(1 - 2h_1\tilde{x}_1 + 4h_1^2\tilde{x}_1^2 + O^{(3+)}\right)^{-1}\left(-\tilde{x}_1 + (2 - h_1)\,\tilde{x}_1^2 + 4h_1\tilde{x}_1^3 + 2h_1^2\tilde{x}_1^4\right) \\
&= -\tilde{x}_1 + (2 + h_1)\,\tilde{x}_1^2 + O^{(3+)} \qquad \text{(I.133)}
\end{aligned}
$$

So, to eliminate the coefficient of the $\tilde{x}_1^2$ term, we need to pick

$$h_1 = -2 \tag{I.134}$$

which gives us our normal form, to quadratic order, of

$$\dot{\tilde{x}}_1 = -\tilde{x}_1 + O^{(3+)} \tag{I.135}$$

◁

Now we will take a look at one more example, which illustrates two things: First, how this process works in more than one dimension; and second, how feedback can be used to eliminate terms that would otherwise appear in the normal form. Our example is a simple two-dimensional linearly controllable system with a few strategically placed non-linear terms.

Example. [Normal Form with Control] Look at the simple two-dimensional dynamic system with control

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2 + x_2^2 \\ \dot{x}_2 &= u \end{aligned} \tag{I.136}$$

where x_1 and x_2 are state variables and u is a single control input. We wish to use coordinate transformations and state feedback to eliminate the two quadratic terms, x_1^2 and x_2^2. We start with the coordinate transformation

$$x = \tilde{x} + H\tilde{x}^{(2)} \tag{I.137}$$

where we have defined

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{I.138}$$

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \tag{I.139}$$

$$\tilde{x}^{(2)} = \begin{bmatrix} \tilde{x}_1^2 \\ \tilde{x}_1\tilde{x}_2 \\ \tilde{x}_2^2 \end{bmatrix} \tag{I.140}$$

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \end{bmatrix} \tag{I.141}$$

where the matrix of coefficients H contains unknown constant coefficients which we will determine later. Now, as in the previous examples, we determine the time rate

of change of the dynamic state vector, $\dot{x}$, two ways and equate them. The first way is to plug our transformation (I.137) into our dynamic system equation (I.136), which yields

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \left(\tilde{x} + H\tilde{x}^{(2)} \right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} \tilde{x}_1^2 + \tilde{x}_2^2 + O^{(3+)} \\ 0 \end{bmatrix} \tag{I.142}$$

where we have used the fact that the quadratic part of the coordinate transformation produces only terms of order three and higher when plugged into the quadratic terms in equation I.136. Now the second way is to differentiate the coordinate transformation (I.137) itself, which yields

$$\begin{aligned} \dot{x} &= \dot{\tilde{x}} + H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}}\dot{\tilde{x}} \\ &= \left(I + H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}} \right) \dot{\tilde{x}} \end{aligned} \tag{I.143}$$

(Note that we have deliberately left the term containing the derivative in unexpanded form for compactness, which we will expand later. It does not hurt to expand it now, it only makes the algebra more complicated and harder to understand.) When we put them together, we get

$$\dot{\tilde{x}} = \left(I + H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}} \right)^{-1} \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \left(\tilde{x} + H\tilde{x}^{(2)} \right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} \tilde{x}_1^2 + \tilde{x}_2^2 + O^{(3+)} \\ 0 \end{bmatrix} \right) \tag{I.144}$$

Now we need to calculate the inverse of $\left(I + H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}} \right)$, which we do with a Taylor series expansion

$$\left(I + H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}} \right)^{-1} = \Phi\left(\tilde{x}\right) = \Phi^{(0)}\left(\tilde{x}\right) + \Phi^{(1)}\left(\tilde{x}\right) + \Phi^{(2)}\left(\tilde{x}\right) + \ldots \tag{I.145}$$

where the notation $\Phi^{(0)}\left(\tilde{x}\right)$, etc. indicates what order of the components of the vector $\tilde{x}$ are included in the function. We can calculate each term by multiplying the quantity by its inverse and setting it equal to the identity matrix. We get

$$\left(I + H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}} \right) \Phi\left(\tilde{x}\right) = \left(I + H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}} \right) \left(\Phi^{(0)}\left(\tilde{x}\right) + \Phi^{(1)}\left(\tilde{x}\right) + \Phi^{(2)}\left(\tilde{x}\right) + \ldots \right)$$

$$
\begin{aligned}
&= \Phi^{(0)}(\tilde{x}) + \left(\Phi^{(1)}(\tilde{x}) + H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}}\Phi^{(0)}(\tilde{x})\right) \\
&\quad + \left(\Phi^{(2)}(\tilde{x}) + H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}}\Phi^{(1)}(\tilde{x})\right) + \ldots \\
&= I
\end{aligned} \tag{I.146}
$$

which gives

$$
\Phi^{(0)}(\tilde{x}) = I \tag{I.147}
$$

$$
\Phi^{(1)}(\tilde{x}) = -H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}} \tag{I.148}
$$

$$
\Phi^{(2)}(\tilde{x}) = \left(-H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}}\right)^2 \tag{I.149}
$$

$$
\vdots
$$

Plugging

$$
\left(I + H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}}\right)^{-1} = I - H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}} + \left(-H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}}\right)^2 + O^{(3+)} \tag{I.150}
$$

back into equation I.144 gives

$$
\begin{aligned}
\dot{\tilde{x}} &= \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\tilde{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u\right) \\
&+ \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}H\tilde{x}^{(2)} - H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\tilde{x} + \begin{bmatrix} \tilde{x}_1^2 + \tilde{x}_2^2 \\ 0 \end{bmatrix}\right) \\
&- \left(H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}u\right) + O^{(3+)}
\end{aligned} \tag{I.151}
$$

As we will see in later chapters, determining the two quadratic order terms is known as solving the "homological equation", a term we will define later. Right now though, we need to define what we mean by $\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}}$, and then make some choices about which terms to cancel by our choice of the components of H. We have

$$
\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}} = \frac{\partial}{\partial \tilde{x}}\begin{bmatrix} \tilde{x}_1^2 \\ \tilde{x}_1\tilde{x}_2 \\ \tilde{x}_2^2 \end{bmatrix} = \begin{bmatrix} 2\tilde{x}_1 & 0 \\ \tilde{x}_2 & \tilde{x}_1 \\ 0 & 2\tilde{x}_2 \end{bmatrix} \tag{I.152}
$$

and choosing to cancel all of the second quadratic order term in equation I.151 (the term multiplied by u) except for the bottom row, we have

$$H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}}\begin{bmatrix} 0 \\ 1 \end{bmatrix} u = \begin{bmatrix} h_{12}\tilde{x}_1 + 2h_{13}\tilde{x}_2 \\ h_{22}\tilde{x}_1 + 2h_{23}\tilde{x}_2 \end{bmatrix} u \tag{I.153}$$

To cancel the top row term, $h_{12}\tilde{x}_1 + 2h_{13}\tilde{x}_2$, for all values of $\tilde{x}_1$ and $\tilde{x}_2$ requires that $h_{12} = h_{13} = 0$. Using this fact, and plugging into the first quadratic order term in equation I.151 gives

$$\begin{aligned} &\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} H\tilde{x}^{(2)} - H\frac{\partial \tilde{x}^{(2)}}{\partial \tilde{x}}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\tilde{x} + \begin{bmatrix} \tilde{x}_1^2 + \tilde{x}_2^2 \\ 0 \end{bmatrix} \\ = &\begin{bmatrix} (h_{21}+1)\,\tilde{x}_1^2 + (h_{22} - 2h_{11})\,\tilde{x}_1\tilde{x}_2 + (h_{23}+1)\,\tilde{x}_2^2 \\ -2h_{21}\tilde{x}_1\tilde{x}_2 - h_{22}\tilde{x}_2^2 \end{bmatrix} \end{aligned} \tag{I.154}$$

Now, we have five terms to cancel, and only four coefficients left: h_{11}, h_{21}, h_{22} and h_{23}. So, we take out as many terms as we can by setting $h_{11} = 0$, $h_{21} = -1$, $h_{22} = 0$ and $h_{23} = -1$ which leaves us with the system

$$\dot{\tilde{x}} = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\tilde{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\right) + \left(\begin{bmatrix} 0 \\ 2\tilde{x}_1\tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2\tilde{x}_2 \end{bmatrix} u\right) + O^{(3+)} \tag{I.155}$$

Finally, we have one last trick up our sleeve. By using feedback, and setting

$$v = (1 - 2\tilde{x}_2)\,u + 2\tilde{x}_1\tilde{x}_2 \tag{I.156}$$

we get the final quadratic normal form of our original control system, which is

$$\dot{\tilde{x}} = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\tilde{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v\right) + O^{(3+)} \tag{I.157}$$

Our full coordinate transformation matrix is

$$H = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} \tag{I.158}$$

which yields a quadratic transformation of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\tilde{x}_1^2 - \tilde{x}_2^2 \end{bmatrix} \tag{I.159}$$

Although all the quadratic terms were able to be eliminated in this example, that is not true in general, particularly for higher dimensional systems, and systems with linearly uncontrollable components of the state vector. We will see in later chapters which terms can be eliminated and which terms remain for the general case. ◁

II. MANIPULATING THE ORIGIN OF COORDINATES

A. PURPOSE OF THIS CHAPTER

In this chapter we will consider how to take an arbitrary affine single-input control system of the form

$$\dot{\breve{x}} = \breve{f}(\breve{x}, \breve{\mu}) + \breve{g}(\breve{x}, \breve{\mu})\,\breve{u} \tag{II.1}$$

and translate the origin of coordinates to achieve a new system of the form

$$\dot{x} = f(x, \mu) + g(x, \mu)\,u \tag{II.2}$$

such that $f(0,0) = 0$. Although this is a trivial coordinate translation if the equilibrium point of interest and the point of bifurcation are known, it is less clear how to proceed for an arbitrary system. Therefore, finding the equilibrium point of interest at the point of bifurcation is the subject of this chapter.

B. TRIMMING THE SYSTEM: FINDING THE CONTROL INPUT NEEDED FOR AN EQUILIBRIUM POINT

Consider again the dynamic control system given by equation II.1. We would like to find a way to put this equation into the form of equation II.2. We start by trimming the system, that is, finding the control input needed to achieve an equilibrium point for our system.

Denote an equilibrium point by $\breve{x}^*$ and the trim control input needed to achieve that equilibrium point by $\breve{u}^*$. Plugging into equation II.1, we get

$$0 = \breve{f}(\breve{x}^*, \breve{\mu}) + \breve{g}(\breve{x}^*, \breve{\mu})\,\breve{u}^* \tag{II.3}$$

where we have used the fact that, by definition, $\dot{\breve{x}} \equiv 0$ for $\breve{x} = \breve{x}^*$ at an equilibrium point. Now, in general we would like to solve equation II.3 for $\breve{x}^*$ in terms of $\breve{u}^*$ and $\breve{\mu}$.

However, that is a very difficult problem, which often does not have a unique solution. For example, consider an aircraft undergoing a flight test of its static stability, as discussed in [Ref. 11]. The test consists of trimming the aircraft over a range of airspeeds, and measuring the trim position of the longitudinal flight control. An aircraft is considered to have neutral static stability if the measured trim position of the longitudinal control is constant for different trimmed airspeeds. Thus, for an aircraft under these conditions, picking a trim position of the longitudinal control and solving for a unique equilibrium airspeed is impossible. However, there is another way to approach the problem, and that is to express the trim control $\breve{u}^*$ as a function of $\breve{x}^*$ and $\breve{\mu}$. That is, if we can somehow find an equilibrium point, the chances are good that we can find a value of the trim control $\breve{u}^*$ which maintains the system there. Writing equation II.3 by components gives us

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \breve{f}_1(\breve{x}^*, \breve{\mu}) + \breve{g}_1(\breve{x}^*, \breve{\mu})\,\breve{u}^* \\ \vdots \\ \breve{f}_n(\breve{x}^*, \breve{\mu}) + \breve{g}_n(\breve{x}^*, \breve{\mu})\,\breve{u}^* \end{bmatrix} \tag{II.4}$$

which allows us to solve for the trim value $\breve{u}^*(\breve{x}^*, \breve{\mu})$ as

$$\breve{u}^*(\breve{x}^*, \breve{\mu}) = -\frac{\breve{f}_1(\breve{x}^*, \breve{\mu})}{\breve{g}_1(\breve{x}^*, \breve{\mu})} = \cdots = -\frac{\breve{f}_n(\breve{x}^*, \breve{\mu})}{\breve{g}_n(\breve{x}^*, \breve{\mu})} \tag{II.5}$$

Picking any of the component equations which are convenient, say the k^{th} component, gives us

$$\breve{u}^*(\breve{x}^*, \breve{\mu}) = -\frac{\breve{f}_k(\breve{x}^*, \breve{\mu})}{\breve{g}_k(\breve{x}^*, \breve{\mu})} \tag{II.6}$$

where it is assumed that $\breve{g}_k(\breve{x}^*, \breve{\mu}) \neq 0$. So, if we can somehow find an equilibrium point $\breve{x}^*$, equation II.6 will give us the steady state trim value of the control input $\breve{u}^*$ needed to maintain our control system there for a particular value of $\breve{\mu}$. However, for different values of $\breve{\mu}$ we will get different values of the trim control $\breve{u}^*$, so it will also be important to discover which value of $\breve{\mu}$ to plug in. (Note that when $\breve{g}_k(\breve{x}^*, \breve{\mu}) = 0$ for all values of k, the steady-state control input $\breve{u}^*(\breve{x}^*, \breve{\mu})$ is arbitrary, and will be decided by other considerations. Note also that even when $\breve{g}_k(\breve{x}^*, \breve{\mu}) \neq 0$, an equilibrium point

defines a control input, but not necessarily vice-versa, as mentioned earlier. That is, a given steady-state control input $\breve{u}^* (\breve{x}^*, \breve{\mu})$ may trim the system at more than one equilibrium point.) Finally, it should be noted that the above analysis is completely dependent on the existence of an equilibrium point, and there is no guarantee at this point in the development that such an equilibrium point exists, with or without steady-state control input. Finding the equilibrium set, if it exists, is the subject of the next section. But first, we illustrate the results of this section with an example.

Example. [Trim Control Input] Consider the affine single-input control system

$$\begin{aligned} \dot{\breve{x}}_1 &= \breve{\mu}_1 \breve{x}_2 + \breve{x}_1^2 + \breve{x}_1^2 \breve{u} \\ \dot{\breve{x}}_2 &= -\breve{x}_2^3 + 3\breve{u} \end{aligned} \tag{II.7}$$

where we can define

$$\breve{f}(\breve{x}, \breve{\mu}) = \begin{bmatrix} \breve{\mu}_1 \breve{x}_2 + \breve{x}_1^2 \\ -\breve{x}_2^3 \end{bmatrix} \tag{II.8}$$

and

$$\breve{g}(\breve{x}, \breve{\mu}) = \begin{bmatrix} \breve{x}_1^2 \\ 3 \end{bmatrix} \tag{II.9}$$

We would like to find the control input $\breve{u}^* (\breve{x}, \breve{\mu})$ which trims the system II.7 at an equilibrium point. Picking the second component of equations II.8 and II.9, and plugging into equation II.6 gives

$$\breve{u}^* (\breve{x}, \breve{\mu}) = -\frac{\breve{f}_2 (\breve{x}^*, \breve{\mu})}{\breve{g}_2 (\breve{x}^*, \breve{\mu})} = \frac{\breve{x}_2^{*3}}{3} \tag{II.10}$$

(where we could have also chosen the first component and found $\breve{u}^* (\breve{x}, \breve{\mu}) = -\frac{\breve{f}_1(\breve{x}^*,\breve{\mu})}{\breve{g}_1(\breve{x}^*,\breve{\mu})} = -\frac{\breve{\mu}_1 \breve{x}_2^* + \breve{x}_1^{*2}}{\breve{x}_1^{*2}}$, so long as $\breve{x}_1^* \neq 0$). But, we do not as yet have any guarantee that an equilibrium point $\breve{x}^*$ exists. We now turn to how to find it. ◁

C. THE EQUILIBRIUM SET

Now that we have algebraically determined the steady-state control input $\breve{u}^* (\breve{x}^*, \breve{\mu})$ needed to trim our system at an equilibrium point, $\breve{x}^*$, given that the

vector of parameters has the value $\breve{\mu}$, we need to solve for the equilibrium set of our system. That is, for all the different possible values of the vector of parameters, $\breve{\mu}$, what are all the different possible equilibrium points, $\breve{x}^*$ for our trimmed system? In general, this is not an easy question to answer, nor will we answer it here. What we will do is to outline the general approach and illustrate the method with examples.

1. Finding the Equilibrium Set

To find the equilibrium set for our system, we can proceed as follows. If we plug the trim control from equation II.6 into equation II.3, and simplify component by component, we end up with $n-1$ independent algebraic equations (the k^{th} component equation drops out):

$$0 = \breve{f}_1(\breve{x}^*, \breve{\mu})\,\breve{g}_k(\breve{x}^*, \breve{\mu}) - \breve{f}_k(\breve{x}^*, \breve{\mu})\,\breve{g}_1(\breve{x}^*, \breve{\mu}) \tag{II.11}$$

$$\vdots$$

$$0 = \breve{f}_{k-1}(\breve{x}^*, \breve{\mu})\,\breve{g}_k(\breve{x}^*, \breve{\mu}) - \breve{f}_k(\breve{x}^*, \breve{\mu})\,\breve{g}_{k-1}(\breve{x}^*, \breve{\mu}) \tag{II.12}$$

$$0 = 0 \tag{II.13}$$

$$0 = \breve{f}_{k+1}(\breve{x}^*, \breve{\mu})\,\breve{g}_k(\breve{x}^*, \breve{\mu}) - \breve{f}_k(\breve{x}^*, \breve{\mu})\,\breve{g}_{k+1}(\breve{x}^*, \breve{\mu}) \tag{II.14}$$

$$\vdots$$

$$0 = \breve{f}_n(\breve{x}^*, \breve{\mu})\,\breve{g}_k(\breve{x}^*, \breve{\mu}) - \breve{f}_k(\breve{x}^*, \breve{\mu})\,\breve{g}_n(\breve{x}^*, \breve{\mu}) \tag{II.15}$$

Now, finding the set of all $\breve{x}^*$ and $\breve{\mu}$ which globally satisfy these equations, if such a set exists, may be a very difficult task. However, there is one thing we can say about the general equilibrium set: Because we have $n-1$ equations in $n+r$ unknowns (n unknown components of $\breve{x}^*$, and r unknown components of $\breve{\mu}$), there will be at least $r+1$ free variables if the equilibrium set exists at all. We will make the assumption, which is often justified for engineering systems, that all of the components of $\breve{\mu}$ and one of the components of $\breve{x}^*$ can be chosen as free variables. We illustrate with an example.

Example. [Equilibrium Set] Let's continue the previous example. Consider again the affine single-input control system

$$\begin{aligned}\dot{\breve{x}}_1 &= \breve{\mu}_1\breve{x}_2 + \breve{x}_1^2 + \breve{x}_1^2\breve{u} \\ \dot{\breve{x}}_2 &= -\breve{x}_2^3 + 3\breve{u}\end{aligned} \tag{II.16}$$

where we have determined our steady-state (trim) control input to be

$$\breve{u}^*\left(\breve{x}^*, \breve{\mu}\right) = \frac{\breve{x}_2^{*3}}{3} \tag{II.17}$$

We would like to find the equilibrium set $\breve{x}^*$ for the system II.16 Plugging equation II.17 into II.16, we find that we have one independent algebraic equation in three unknowns, i.e.

$$0 = \breve{\mu}_1\breve{x}_2 + \breve{x}_1^2 + \breve{x}_1^2\frac{\breve{x}_2^{*3}}{3} \tag{II.18}$$

$$0 = 0 \tag{II.19}$$

where the second equation has fallen out. This system is solvable for $\breve{x}_1^*$ if we choose $\breve{x}_2^*$ and $\breve{\mu}_1$ as our free variables, i.e.

$$\breve{x}_1^* = \pm\sqrt{\frac{-\breve{\mu}_1\breve{x}_2^*}{\left(1 + \frac{\breve{x}_2^{*3}}{3}\right)}} \tag{II.20}$$

So, equation II.20 gives a solution for the equilibrium set for our system. However, although an equilibrium set does exist for this system, an equilibrium point does not necessarily exist for all the possible values of the free variables $\breve{\mu}_1$ and $\breve{x}_2^*$, since (as one example) when both $\breve{\mu}_1 > 0$ and $\breve{x}_2^* > 0$ no real values of $\breve{x}_1^*$ exist. ◃

2. Control Inputs and the Equilibrium Set

In the above section we dealt with finding the equilibrium set after the dynamic control system had been trimmed. We framed our search for the equilibrium set in terms of the free variables, which we assumed were the components of the vector of parameters, $\breve{\mu}$, and one of the state variables at the equilibrium point, $\breve{x}_k^*$. Then, we

proceeded to search for a solution to the remaining components of the state vector at equilibrium in terms of $\breve{\mu}$ and $\breve{x}_k^*$, assuming that these two could take on any values we chose. In reality, of course, it is not that simple. Engineering systems do not operate for arbitrary values of the state variables and parameters, but only over a restricted domain. Also, the "free variable" component of the the state vector, $\breve{x}_k^*$, may not actually be a free variable under our control. Rather, over a properly restricted domain, our actual free variable is the control input, $\breve{u}$. That is, using the control $\breve{u}$, we must first establish the system at the desired equilibrium point $\breve{x}^*$, then apply the trim control $\breve{u}^*$ necessary to maintain the system there. Thus, the assumption that a component of $\breve{x}^*$ is a free variable is only justified under the circumstance that our control input $\breve{u}$ is capable of achieving the desired equilibrium point to begin with. In subsequent sections, we will continue to talk about a free variable component of the equilibrium state vector because it makes the mathematical manipulations easier. However, in all cases this should be understood to mean that a control input $\breve{u}$ capable of establishing the system at the equilibrium point has been used to get the system there.

D. FINDING THE BIFURCATION POINTS

Now, assuming that an equilibrium set exists, and that we can trim the system there, we would like to find the points of bifurcation. That is, for what values of $\breve{\mu}$ does a qualitative change in the equilibrium set occur? Here we define the term "qualitative change" as a change in the structure of the trajectories of a system, such as a change in the number of equilibrium points, a change in the stability of an equilibrium point, the generation or destruction of periodic solutions (limit cycles), etc. We note that, in all cases, a linearization around an equilibrium point having at least one zero real-part eigenvalue is a necessary (but not sufficient) condition for a bifurcation to occur. In general, this is not an easy problem to solve, nor will we solve the general case here. What we will do is to outline two useful approaches to

the problem and illustrate the method with examples.

1. Change in the Number of Equilibrium Points

One easy, but not comprehensive, check for a bifurcation point is to determine whether equilibrium points are being created or destroyed at a given value of $\breve{\mu}$. Since we have assumed that one component of $\breve{x}^*$ and all components of $\breve{\mu}$ are free variables, equations II.11 through II.15 can be examined to find those values of $\breve{\mu}$ for which the number of equilibrium points change. We illustrate with an example.

Example. [Change in the Number of Equilibrium Points] Continuing the previous example, we examine the solution for the equilibrium set, to see if there is a value of $\breve{\mu}_1$ which causes the number of equilibrium points to change. We have

$$\breve{x}_1^* = \pm\sqrt{\frac{-\breve{\mu}_1 \breve{x}_2^*}{\left(1+\frac{\breve{x}_2^{*3}}{3}\right)}} \tag{II.21}$$

Now, for any value of the free variable $\breve{x}_2^*$, the sign of the quantity $\frac{\breve{x}_2^*}{\left(1+\frac{\breve{x}_2^{*3}}{3}\right)}$ determines the allowable values of $\breve{\mu}_1$ such that the equilibrium set exists. When

$$\frac{\breve{x}_2^*}{\left(1+\frac{\breve{x}_2^{*3}}{3}\right)} > 0 \tag{II.22}$$

two equilibrium points exist for $\breve{\mu}_1 < 0$, one equilibrium point exists for $\breve{\mu}_1 = 0$, and no equilibrium points exist for $\breve{\mu}_1 > 0$. When

$$\frac{\breve{x}_2^*}{\left(1+\frac{\breve{x}_2^{*3}}{3}\right)} < 0 \tag{II.23}$$

no equilibrium points exist for $\breve{\mu}_1 < 0$, one equilibrium point exists for $\breve{\mu}_1 = 0$, and two equilibrium points exist for $\breve{\mu}_1 > 0$. Finally, when

$$\frac{\breve{x}_2^*}{\left(1+\frac{\breve{x}_2^{*3}}{3}\right)} = 0 \tag{II.24}$$

one equilibrium point exists regardless of the value of $\breve{\mu}_1$. So, in this example, the value $\breve{\mu}_1^* = 0$, a constant, is the point of bifurcation. In the general case however, the bifurcation point will be a function of the free variable component of $\breve{x}^*$. ◁

2. Change in the Stability of an Equilibrium Point

To examine a change in the stability of an equilibrium point, we need to evaluate the eigenvalues of the Jacobian matrix of our system at that equilibrium point. We desire to find those values of $\breve{\mu}$ for which one or several eigenvalues have a zero real part — this indicates the point at which stability changes. To do this we need to examine our original control system trimmed at the equilibrium point of interest, that is, when steady-state control which achieves that equilibrium point is applied. We need to examine the Jacobian matrix of the system

$$\dot{\breve{x}} = \breve{f}\left(\breve{x}, \breve{\mu}\right) + \breve{g}\left(\breve{x}, \breve{\mu}\right) \breve{u}^{*}\left(\breve{x}^{*}, \breve{\mu}\right) \tag{II.25}$$

around the point $\breve{x} = \breve{x}^{*}$, and determine those values of $\breve{\mu}$ for which the real part of one or several eigenvalues is zero. If we let J be the Jacobian matrix of this system, and recognizing that the value $\breve{u}^{*}\left(\breve{x}^{*}, \breve{\mu}\right)$ is a constant, we have

$$J = \left(\frac{\partial \breve{f}\left(\breve{x}, \breve{\mu}\right)}{\partial \breve{x}}\right)_{\breve{x}=\breve{x}^{*}} + \left(\frac{\partial \breve{g}\left(\breve{x}, \breve{\mu}\right)}{\partial \breve{x}}\right)_{\breve{x}=\breve{x}^{*}} \breve{u}^{*}\left(\breve{x}^{*}, \breve{\mu}\right) \tag{II.26}$$

We can determine the eigenvalues λ of the Jacobian matrix J by solving the equation

$$\det\left(\lambda I - J\right) = 0 \tag{II.27}$$

and the points of bifurcation $\breve{\mu} = \breve{\mu}^{*}$ are found when the real part of any eigenvalue is equal to zero. We illustrate with an example.

Example. [Change in the Stability of an Equilibrium Point] Continuing with the system and steady-state control input from the second example, our dynamic system trimmed at an equilibrium point $\breve{x}^{*}$ is

$$\begin{aligned} \dot{\breve{x}}_1 &= \breve{\mu}_1 \breve{x}_2 + \breve{x}_1^2 + \breve{x}_1^2 \breve{u}^{*}\left(\breve{x}^{*}, \breve{\mu}\right) \\ \dot{\breve{x}}_2 &= -\breve{x}_2^3 + 3\breve{u}^{*}\left(\breve{x}^{*}, \breve{\mu}\right) \end{aligned} \tag{II.28}$$

where

$$\breve{u}^{*}\left(\breve{x}^{*}, \breve{\mu}\right) = \frac{\breve{x}_2^{*3}}{3} \tag{II.29}$$

is a constant control input, since we are assumed to be trimmed at the equilibrium point defined by $\breve{x}_2^*$. Calculating the components of the Jacobian matrix we have

$$\left(\frac{\partial \breve{f}\left(\breve{x}, \breve{\mu}\right)}{\partial \breve{x}}\right)_{\breve{x}=\breve{x}^*} = \begin{bmatrix} 2\breve{x}_1^* & \breve{\mu}_1 \\ 0 & -3\breve{x}_2^{*2} \end{bmatrix} \tag{II.30}$$

and

$$\left(\frac{\partial \breve{g}\left(\breve{x}, \breve{\mu}\right)}{\partial \breve{x}}\right)_{\breve{x}=\breve{x}^*} = \begin{bmatrix} 2\breve{x}_1^* & 0 \\ 0 & 0 \end{bmatrix} \tag{II.31}$$

and putting the pieces together yields

$$J = \begin{bmatrix} 2\breve{x}_1^*\left(1 + \frac{\breve{x}_2^{*3}}{3}\right) & \breve{\mu}_1 \\ 0 & -3\breve{x}_2^{*2} \end{bmatrix} \tag{II.32}$$

where we have plugged in equation II.29 for $\breve{u}^*\left(\breve{x}^*, \breve{\mu}\right)$. Now, the equation to solve for the eigenvalues of J is

$$\det\left(\lambda I - J\right) = \det \begin{bmatrix} \lambda - 2\breve{x}_1^*\left(1 + \frac{\breve{x}_2^{*3}}{3}\right) & -\breve{\mu}_1 \\ 0 & \lambda + 3\breve{x}_2^{*2} \end{bmatrix} = 0 \tag{II.33}$$

which is

$$\left(\lambda - 2\breve{x}_1^*\left(1 + \frac{\breve{x}_2^{*3}}{3}\right)\right)\left(\lambda + 3\breve{x}_2^{*2}\right) = 0 \tag{II.34}$$

This is true when

$$\lambda = 2\breve{x}_1^*\left(1 + \frac{\breve{x}_2^{*3}}{3}\right) \tag{II.35}$$

or

$$\lambda = -3\breve{x}_2^{*2} \tag{II.36}$$

Now, we look at all possible equilibrium points where we could trim our system. Since $\breve{x}_2^*$ is a free variable of our equilibrium set, it is possible to trim the system to an equilibrium point such that $\breve{x}_2^* \neq 0$. For such an equilibrium point, the real part of λ can only equal zero when $\breve{x}_1^* = 0$. Looking at equation II.20, we see that the only value of $\breve{\mu}_1$ which causes $\breve{x}_1^* = 0$ is $\breve{\mu}_1 = 0$. Thus $\breve{\mu}_1 = 0$ is our bifurcation point. (Note that when $\breve{x}_2^* = 0$, we have λ equal to zero regardless of the value of the parameter

$\breve{\mu}_1$. Thus, no bifurcation occurs along this narrow subset of the equilibrium set. For $\breve{x}_2^* = -\sqrt[3]{3}$, we also have λ equal to zero regardless of the value of the parameter $\breve{\mu}_1$. However, the value $\breve{x}_2^* = -\sqrt[3]{3}$ does not indicate a bifurcation; rather, there is no equilibrium set at this value, except when $\breve{\mu}_1 = 0$). ◁

E. TRANSLATING THE ORIGIN OF COORDINATES

Because of the general nature of our equations, we can say very little about the existence of a global equilibrium set, nor do we have any guarantees that a general system can be trimmed. However, as was mentioned earlier, engineering systems operate in a restricted domain, not everywhere. In a local region of operation, locating the equilibrium points and bifurcation points is often much easier than trying to determine them globally. Plus, engineering systems tend to be reasonably well behaved, so it is a reasonable assumption that we will be able to find a trim condition. So, with these points in mind, we will make the following assumptions:

- An equilibrium point of interest, $\breve{x}^*$, exists and we have found it. In particular, we have chosen a particular value for that component of $\breve{x}^*$ which is acting as a free variable.
- We can solve for the trim value of the control $\breve{u}^*$ and the value of the bifurcation point $\breve{\mu}$.

Making these assumptions, let us write our state vector, $\breve{x}$, our vector of parameters, $\breve{\mu}$, and our control input, $\breve{u}$, as perturbations away from the equilibrium point/point of bifurcation we have chosen. We get

$$\breve{x} = \breve{x}^* + x \tag{II.37}$$

$$\breve{\mu} = \breve{\mu}^* + \mu \tag{II.38}$$

$$\breve{u} = \breve{u}^* + u \tag{II.39}$$

Plugging these into equation II.1 gives

$$\dot{\breve{x}} = \breve{f}\left(\breve{x}^* + x, \breve{\mu}^* + \mu\right) + \breve{g}\left(\breve{x}^* + x, \breve{\mu}^* + \mu\right)\left(\breve{u}^*\left(\breve{x}^*, \breve{\mu}^*\right) + u\right) \tag{II.40}$$

Since, by definition, $\breve{x}^*$ is a constant with zero derivative, we can rewrite equation II.40 as

$$\dot{x} = f(x, \mu) + g(x, \mu)\, u \tag{II.41}$$

with

$$f(x, \mu) = \breve{f}(\breve{x}^* + x, \breve{\mu}^* + \mu) + \breve{g}(\breve{x}^* + x, \breve{\mu}^* + \mu)\, \breve{u}^*(\breve{x}^*, \breve{\mu}^*) \tag{II.42}$$

$$g(x, \mu) = \breve{g}(\breve{x}^* + x, \breve{\mu}^* + \mu) \tag{II.43}$$

and where

$$f(0, 0) = 0 \tag{II.44}$$

Equation II.41 is the basis for all of the further transformations which will be performed in this dissertation. In that sense, equation II.41 is the beginning of this dissertation. As before, we illustrate with an example.

Example. [Translating the Origin] Continuing our example system, we have

$$\begin{aligned} \dot{\breve{x}}_1 &= \breve{\mu}_1 \breve{x}_2 + \breve{x}_1^2 + \breve{x}_1^2 \breve{u} \\ \dot{\breve{x}}_2 &= -\breve{x}_2^3 + 3\breve{u} \end{aligned} \tag{II.45}$$

where we make the assumption that have used external engineering considerations (not shown here) to choose the free variable component of $\breve{x}^*$ as

$$\breve{x}_2^* = 1 \tag{II.46}$$

In the second example, we determined our steady-state control input. Plugging equation II.46 into equation II.17 we get

$$\breve{u}^*(\breve{x}^*, \breve{\mu}) = \frac{1}{3} \tag{II.47}$$

In the third and fourth examples, we found that the bifurcation point for our system was

$$\breve{\mu}_1^* = 0 \tag{II.48}$$

which lets us use equation II.20 for the equilibrium set. Plugging equations II.46 and II.48 into equation II.20 gives

$$\breve{x}_1^* = 0 \tag{II.49}$$

So, translating our origin of coordinates by the rules

$$\begin{aligned}
\breve{x}_1 &= x_1 && \text{(II.50)}\\
\breve{x}_2 &= 1 + x_2 && \text{(II.51)}\\
\breve{\mu}_1 &= \mu_1 && \text{(II.52)}\\
\breve{u} &= \frac{1}{3} + u && \text{(II.53)}
\end{aligned}$$

and plugging into equation II.45, gives a new control system

$$\begin{aligned}
\dot{x}_1 &= \mu + \mu x_2 + \frac{4}{3}x_1^2 + x_1^2 u && \text{(II.54)}\\
\dot{x}_2 &= -3x_2 - 3x_2^2 - x_2^3 + 3u && \text{(II.55)}
\end{aligned}$$

where our chosen equilibrium point, at the point of bifurcation, is the origin of coordinates. ◁

III. LINEAR NORMAL FORM

A. ROADMAP: THE BIG PICTURE

1. Results of Previous Chapters

In Chapter II we showed that the origin of coordinates for any affine control system

$$\dot{\breve{x}} = \breve{f}\left(\breve{x}, \breve{\mu}\right) + \breve{g}\left(\breve{x}, \breve{\mu}\right)\breve{u} \tag{III.1}$$

could be translated to an equilibrium point $\breve{x}^*$ at the point of bifurcation $\breve{\mu}^*$ using the trim control $\breve{u}^*$, making the reasonable assumptions that an equilibrium set exists, that the system can be trimmed, and that a bifurcation occurs. The translated affine control system has the form

$$\dot{x} = f\left(x, \mu\right) + g\left(x, \mu\right)u \tag{III.2}$$

with

$$f\left(0, 0\right) = 0 \tag{III.3}$$

2. Purpose of this Chapter

In this chapter, we begin the process of simplifying a system in the form of III.2 by applying coordinate transformations and state feedback. This chapter considers how to simplify the linear terms in our control system, and Chapter V considers how to simplify the quadratic terms. Taken all together, the methods of Chapters II, III and V will produce a system which is in quadratic normal form.

In this chapter we show how to take an affine single-input control system of the form of III.2 and transform it into a system of the form

$$\begin{aligned}\begin{bmatrix}\dot{\mu}\\ \dot{\tilde{z}}\\ \dot{\tilde{w}}\\ \dot{\tilde{y}}\end{bmatrix} &= \begin{bmatrix}0 & 0 & 0 & 0\\ F_\mu & F_z & 0 & 0\\ 0 & 0 & F_w & 0\\ 0 & 0 & 0 & A\end{bmatrix}\begin{bmatrix}\mu\\ \tilde{z}\\ \tilde{w}\\ \tilde{y}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ B\end{bmatrix}\tilde{u} \\ &+ \tilde{f}^{(2)}\left(\mu, \tilde{z}, \tilde{w}, \tilde{y}\right) + \tilde{g}^{(1)}\left(\mu, \tilde{z}, \tilde{w}, \tilde{y}\right)\tilde{u}\end{aligned} \tag{III.4}$$

$$+ \quad \tilde{f}^{(3)}(\mu, \tilde{z}, \tilde{w}, \tilde{y}) + \tilde{g}^{(2)}(\mu, \tilde{z}, \tilde{w}, \tilde{y})\,\tilde{u} + O^{(4+)}$$

We accomplish this with the following steps:

- Expand equation III.2 in a multi-variable Taylor series around the origin.
- Apply a linear similarity coordinate transformation to the Taylor series expansion to simplify the linear terms.
- Apply linear state feedback to the transformed system produce further simplification of the linear terms.

B. MULTI-VARIABLE TAYLOR SERIES EXPANSION

Consider again the dynamic control system given in equation III.2. We would like to examine the dynamics of our system in the local vicinity of the equilibrium point of interest, around the point of bifurcation. Since our origin of coordinates is already at the equilibrium point of interest and at the point of bifurcation, we only need to expand our system in a Taylor series around the origin to obtain the local dynamics. By including the vector of parameters μ as a variable in our Taylor series expansion, we will also capture the local bifurcation phenomena. We start by defining some notation to make multi-variable Taylor series expansions easier to manipulate.

1. Notation Conventions

Conventional multi-variable Taylor series notation quickly becomes cumbersome when dealing with vector-valued functions. For example, expanding the simple three-component system with two independent variables

$$f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ f_3(x_1, x_2) \end{bmatrix} \tag{III.5}$$

to only second order in a multi-variable Taylor series expansion around the origin in conventional notation yields three component equations of the form

$$f_i(x_1, x_2) \quad = \quad f_i(0,0) + \left(\frac{\partial f_i}{\partial x_1}\right)_{x=0} x_1 + \left(\frac{\partial f_i}{\partial x_2}\right)_{x=0} x_2 \tag{III.6}$$

$$+ \frac{1}{2!}\left(\frac{\partial^2 f_i}{\partial x_1^2}\right)_{x=0} x_1^2 + \frac{1}{2!}\left(2\frac{\partial^2 f_i}{\partial x_1 \partial x_2}\right)_{x=0} x_1 x_2 + \frac{1}{2!}\left(\frac{\partial^2 f_i}{\partial x_2^2}\right)_{x=0} x_2^2 + O^{(3+)}$$

(for $i = 1$ to 3) which is clearly difficult to deal with, even for this simple case. For more complicated systems, and in particular, for systems of arbitrary dimension, conventional notation is so difficult as to be unusable.

However, there are simpler ways to express multi-variable Taylor series expansions for vector valued functions. One of these is the vector/matrix expansion, where the coefficients are grouped in matrices, and the variables are stacked in vectors, all according to the order (linear, quadratic, etc.) of the terms involved. For example, the Taylor series expansion of equation III.5 could be written in vector/matrix form as

$$f(x) = f(0) + L_f \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Q_f \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix} + O^{(3+)} \tag{III.7}$$

where the coefficient matrices are

$$f(0) = \begin{bmatrix} f_1(0,0) \\ f_2(0,0) \\ f_3(0,0) \end{bmatrix} \tag{III.8}$$

$$L_f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix}_{x=0} \tag{III.9}$$

and

$$Q_f = \frac{1}{2!}\begin{bmatrix} \frac{\partial^2 f_1}{\partial x_1^2} & 2\frac{\partial^2 f_1}{\partial x_1 \partial x_2} & \frac{\partial^2 f_1}{\partial x_2^2} \\ \frac{\partial^2 f_2}{\partial x_1^2} & 2\frac{\partial^2 f_2}{\partial x_1 \partial x_2} & \frac{\partial^2 f_2}{\partial x_2^2} \\ \frac{\partial^2 f_3}{\partial x_1^2} & 2\frac{\partial^2 f_3}{\partial x_1 \partial x_2} & \frac{\partial^2 f_3}{\partial x_2^2} \end{bmatrix}_{x=0} \tag{III.10}$$

Another way which is even simpler (called functional order notation), is to merely indicate the order of the terms involved and otherwise leave the system in functional notation. For example, the expansion of equation III.5 can be written in functional

order notation as

$$f(x) = f^{(0)}(x) + f^{(1)}(x) + f^{(2)}(x) + O^{(3+)} \tag{III.11}$$

where the individual term orders are indicated by the superscript value in parentheses, i.e.

$$f^{(0)}(x) = \begin{bmatrix} f_1(0,0) \\ f_2(0,0) \\ f_3(0,0) \end{bmatrix} \tag{III.12}$$

$$f^{(1)}(x) = \begin{bmatrix} \left(\frac{\partial f_1}{\partial x_1}\right)_{x=0} x_1 + \left(\frac{\partial f_1}{\partial x_2}\right)_{x=0} x_2 \\ \left(\frac{\partial f_2}{\partial x_1}\right)_{x=0} x_1 + \left(\frac{\partial f_2}{\partial x_2}\right)_{x=0} x_2 \\ \left(\frac{\partial f_3}{\partial x_1}\right)_{x=0} x_1 + \left(\frac{\partial f_3}{\partial x_2}\right)_{x=0} x_2 \end{bmatrix} \tag{III.13}$$

$$f^{(2)}(x) = \begin{bmatrix} \frac{1}{2!}\left(\frac{\partial^2 f_1}{\partial x_1^2}\right)_{x=0} x_1^2 + \frac{1}{2!}\left(2\frac{\partial^2 f_1}{\partial x_1 \partial x_2}\right)_{x=0} x_1 x_2 + \frac{1}{2!}\left(\frac{\partial^2 f_1}{\partial x_2^2}\right)_{x=0} x_2^2 \\ \frac{1}{2!}\left(\frac{\partial^2 f_2}{\partial x_1^2}\right)_{x=0} x_1^2 + \frac{1}{2!}\left(2\frac{\partial^2 f_2}{\partial x_1 \partial x_2}\right)_{x=0} x_1 x_2 + \frac{1}{2!}\left(\frac{\partial^2 f_2}{\partial x_2^2}\right)_{x=0} x_2^2 \\ \frac{1}{2!}\left(\frac{\partial^2 f_3}{\partial x_1^2}\right)_{x=0} x_1^2 + \frac{1}{2!}\left(2\frac{\partial^2 f_3}{\partial x_1 \partial x_2}\right)_{x=0} x_1 x_2 + \frac{1}{2!}\left(\frac{\partial^2 f_3}{\partial x_2^2}\right)_{x=0} x_2^2 \end{bmatrix} \tag{III.14}$$

Finally, the expansion can be written in tensor notation. However, although tensor notation is very compact, it does not lend itself to ease of computation with the tools available today. Since a focus of this dissertation is the computation of feedback gains to control or stabilize bifurcations, we will not deal with tensors here. Instead, we will concentrate on the vector/matrix form and the functional order form. Both of these methods are formalized in the following lemma.

Lemma B.1 (Notation for Multi-Variable Taylor Series Expansions) *Given a function $\Psi(\xi)$, with $\Psi \in R^s$ and $\xi \in R^t$, the multi-variable Taylor series expansion around the origin can be expressed in vector/matrix form as*

$$\Psi(\xi) = \Psi(0) + L_\Psi \xi + Q_\Psi \xi^{(2)} + C_\Psi \xi^{(3)} + O^{(4+)} \tag{III.15}$$

or in functional order form as

$$\Psi(\xi) = \Psi^{(0)}(\xi) + \Psi^{(1)}(\xi) + \Psi^{(2)}(\xi) + \Psi^{(3)}(\xi) + O^{(4+)} \tag{III.16}$$

where the order of the variable terms is given by the superscript value in parentheses. The matrix of coefficients L_Ψ (the L is for "linear") is known as the Jacobian of

Ψ, *while the matrix of coefficients* Q_Ψ *(the* Q *is for "quadratic") and the matrix of coefficients* C_Ψ *(the* C *is for "cubic") do not have a commonly accepted names. These matrices are given by the formulas*

$$L_\Psi = \begin{bmatrix} \frac{\partial \Psi_1}{\partial \xi_1} & \cdots & \frac{\partial \Psi_1}{\partial \xi_s} \\ & \vdots & \\ \frac{\partial \Psi_s}{\partial \xi_1} & \cdots & \frac{\partial \Psi_s}{\partial \xi_s} \end{bmatrix}_{(\xi=0)} \tag{III.17}$$

$$Q_\Psi = \frac{1}{2!} \begin{bmatrix} \frac{\partial^2 \Psi_1}{\partial \xi_1^2} & 2\frac{\partial^2 \Psi_1}{\partial \xi_1 \partial \xi_2} & \frac{\partial^2 \Psi_1}{\partial \xi_2^2} & \cdots & \frac{\partial^2 \Psi_1}{\partial \xi_s^2} \\ & \vdots & & \vdots & \\ \frac{\partial^2 \Psi_s}{\partial \xi_1^2} & 2\frac{\partial^2 \Psi_s}{\partial \xi_1 \partial \xi_2} & \frac{\partial^2 \Psi_s}{\partial \xi_2^2} & \cdots & \frac{\partial^2 \Psi_s}{\partial \xi_s^2} \end{bmatrix}_{(\xi=0)} \tag{III.18}$$

$$C_\Psi = \frac{1}{3!} \begin{bmatrix} \frac{\partial^3 \Psi_1}{\partial \xi_1^3} & 3\frac{\partial^3 \Psi_1}{\partial \xi_1^2 \partial \xi_2} & 3\frac{\partial^3 \Psi_1}{\partial \xi_1 \partial \xi_2^2} & \frac{\partial^3 \Psi_1}{\partial \xi_2^3} & 3\frac{\partial^3 \Psi_1}{\partial \xi_1^2 \partial \xi_3} & 6\frac{\partial^3 \Psi_1}{\partial \xi_1 \partial \xi_2 \partial \xi_3} & \cdots & \\ & \cdots & 3\frac{\partial^3 \Psi_1}{\partial \xi_2^2 \partial \xi_3} & 3\frac{\partial^3 \Psi_1}{\partial \xi_1 \partial \xi_3^2} & 3\frac{\partial^3 \Psi_1}{\partial \xi_2 \partial \xi_3^2} & \frac{\partial^3 \Psi_1}{\partial \xi_3^3} & \cdots & \frac{\partial^3 \Psi_1}{\partial \xi_s^3} \\ & \vdots & & & & & \vdots & \\ \frac{\partial^3 \Psi_s}{\partial \xi_1^3} & 3\frac{\partial^3 \Psi_s}{\partial \xi_1^2 \partial \xi_2} & 3\frac{\partial^3 \Psi_s}{\partial \xi_1 \partial \xi_2^2} & \frac{\partial^3 \Psi_s}{\partial \xi_2^3} & 3\frac{\partial^3 \Psi_s}{\partial \xi_1^2 \partial \xi_3} & 6\frac{\partial^3 \Psi_s}{\partial \xi_1 \partial \xi_2 \partial \xi_3} & \cdots & \\ & \cdots & 3\frac{\partial^3 \Psi_s}{\partial \xi_2^2 \partial \xi_3} & 3\frac{\partial^3 \Psi_s}{\partial \xi_1 \partial \xi_3^2} & 3\frac{\partial^3 \Psi_s}{\partial \xi_2 \partial \xi_3^2} & \frac{\partial^3 \Psi_s}{\partial \xi_3^3} & \cdots & \frac{\partial^3 \Psi_s}{\partial \xi_s^3} \end{bmatrix}_{(\xi=0)} \tag{III.19}$$

where $L_\Psi \in R^{s \times t}$, $Q_\Psi \in R^{s \times \frac{t(t+1)}{2}}$, *and* $C_\Psi \in R^{s \times \frac{t(t+1)(t+2)}{6}}$. *New notation has also been introduced for the vectors of variables,* $\xi^{(2)} \in R^{\frac{t(t+1)}{2}}$ *and* $\xi^{(3)} \in R^{\frac{t(t+1)(t+2)}{6}}$. *Here* $\xi^{(2)}$ *indicates the vector of all possible quadratic combinations of the components of* ξ, *while* $\xi^{(3)}$ *indicates the vector of all possible cubic combinations of the components of* ξ. *The rule for ordering the elements* $\xi_h\xi_i$ *of the quadratic vector of variables* $\xi^{(2)}$ *is* $\xi_h\xi_i > \xi_j\xi_k$ *if* $i > k$, *or* $h > j$ *if* $i = k$. *Then the smaller elements are stacked on top of the larger elements. The rule for ordering the elements* $\xi_h\xi_i\xi_j$ *of the cubic vector of variables* $\xi^{(3)}$ *is* $\xi_h\xi_i\xi_j > \xi_k\xi_l\xi_m$ *if* $j > m$, *or* $i > l$ *if* $j = m$, *or* $h > k$ *if* $j = m$ *and* $i = l$. *Then the smaller elements are stacked on top of the larger elements. These vectors may be written out as*

$$\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_s \end{bmatrix} \tag{III.20}$$

$$\xi^{(2)} = \begin{bmatrix} \xi_1^2 \\ \xi_1\xi_2 \\ \xi_2^2 \\ \vdots \\ \xi_s^2 \end{bmatrix} \tag{III.21}$$

$$\xi^{(3)} = \begin{bmatrix} \xi_1^3 \\ \xi_1^2\xi_2 \\ \xi_1\xi_2^2 \\ \xi_2^3 \\ \xi_1^2\xi_3 \\ \xi_1\xi_2\xi_3 \\ \xi_2^2\xi_3 \\ \xi_1\xi_3^2 \\ \xi_2\xi_3^2 \\ \xi_3^3 \\ \vdots \\ \xi_s^3 \end{bmatrix} \tag{III.22}$$

Note that in subsequent sections these will be the common methods of Taylor series expansion and will be used interchangeably.

Proof. Perform a conventional multi-variable Taylor series expansion on each of the components of $\Psi(\xi)$. Then group the terms by order, and separate out the coefficients into matrices and the variable combinations into the vectors provided. This yields the results in the lemma. ◁

Now we illustrate our notation with an example.

Example. [Taylor Series Expansion] Find the Taylor series expansion around the origin of the function $f(x)$ given by

$$f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} \sin(2x_1 + x_2) \\ x_1 e^{x_2} \end{bmatrix} \tag{III.23}$$

Now, our lemma tells us that the function $f(x)$ can be expanded around the origin as follows:

$$f(x) = f^{(0)}(x) + f^{(1)}(x) + f^{(2)}(x) + f^{(3)}(x) + O^{(4+)} \tag{III.24}$$

$$= F_0 + L_f \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Q_f \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix} + C_f \begin{bmatrix} x_1^3 \\ x_1^2x_2 \\ x_1x_2^2 \\ x_2^3 \end{bmatrix} + O^{(4)} \tag{III.25}$$

We calculate the vector F_0 and the matrices L_f, Q_f, and C_f by the formulas

$$F_0 = f(0) = \begin{bmatrix} \sin(2x_1 + x_2) \\ x_1 e^{x_2} \end{bmatrix}_{(x=0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{III.26}$$

$$\begin{aligned} L_f &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(x=0)} = \begin{bmatrix} 2\cos(2x_1 + x_2) & \cos(2x_1 + x_2) \\ e^{x_2} & x_1 e^{x_2} \end{bmatrix}_{(x=0)} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned} \tag{III.27}$$

$$\begin{aligned} Q_f &= \frac{1}{2!}\begin{bmatrix} \frac{\partial^2 f_1}{\partial x_1^2} & \frac{\partial^2 f_1}{\partial x_1 \partial x_2} & \frac{\partial^2 f_1}{\partial x_2^2} \\ \frac{\partial^2 f_2}{\partial x_1^2} & \frac{\partial^2 f_2}{\partial x_1 \partial x_2} & \frac{\partial^2 f_2}{\partial x_2^2} \end{bmatrix}_{(x=0)} \\ &= \frac{1}{2}\begin{bmatrix} -4\sin(2x_1 + x_2) & -4\sin(2x_1 + x_2) & -\sin(2x_1 + x_2) \\ 0 & 2e^{x_2} & x_1 e^{x_2} \end{bmatrix}_{(x=0)} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned} \tag{III.28}$$

$$\begin{aligned} C_f &= \frac{1}{3!}\begin{bmatrix} \frac{\partial^3 f_1}{\partial x_1^3} & 3\frac{\partial^3 f_1}{\partial x_1^2 \partial x_2} & 3\frac{\partial^3 f_1}{\partial x_1 \partial x_2^2} & \frac{\partial^3 f_1}{\partial x_2^3} \\ \frac{\partial^3 f_2}{\partial x_1^3} & 3\frac{\partial^3 f_2}{\partial x_1^2 \partial x_2} & 3\frac{\partial^3 f_2}{\partial x_1 \partial x_2^2} & \frac{\partial^3 f_2}{\partial x_2^3} \end{bmatrix}_{(x=0)} \\ &= -\frac{1}{6}\begin{bmatrix} 8\cos(2x_1 + x_2) & 12\cos(2x_1 + x_2) & 6\cos(2x_1 + x_2) & \cos(2x_1 + x_2) \\ 0 & 0 & -3e^{x_2} & -x_1 e^{x_2} \end{bmatrix}_{(x=0)} \\ &= \begin{bmatrix} -\frac{4}{3} & -2 & -1 & -\frac{1}{6} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \end{aligned} \tag{III.29}$$

which gives our answer in vector/matrix form as

$$f(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix} \tag{III.30}$$

$$+ \begin{bmatrix} -\frac{4}{3} & -2 & -1 & -\frac{1}{6} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{bmatrix} + O^{(4+)}$$

or in functional order form as

$$\begin{aligned} f(x) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_1 + x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ x_1 x_2 \end{bmatrix} \\ &+ \begin{bmatrix} -\frac{4}{3}x_1^3 - 2x_1^2 x_2 - x_1 x_2^2 - \frac{1}{6}x_2^3 \\ \frac{1}{2}x_1 x_2^2 \end{bmatrix} + O^{(4+)} \end{aligned} \tag{III.31}$$

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Finally, it is worth asking why we choose to expand our Taylor series this way. First, the notation is very compact, and by using L, Q, and C to stand for linear, quadratic and cubic in the vector/matrix notation, or to indicate the term order inside superscript parentheses in the functional order notation, it is also reasonably intuitive as well. Second, as we will see later on in the chapter, putting the coefficients into matrices allows us to perform block matrix manipulations, which will lead to powerful simplifications. And third, after our transformations and simplifications are complete, the entries inside the matrices will mean something, much as the entries in a diagonal matrix give us the eigenvalues of that matrix. (In fact, the diagonal entries in our transformed linear coefficient matrix will be the eigenvalues of the linear portion of the system.) The entries in the transformed quadratic matrix will tell us whether the bifurcation which occurs is controllable, and will then feed directly into the formulas for the gains necessary to control our system.

2. Taylor Series Expansion of the Control System

Now let's look at the Taylor series expansion of our control system around the origin, where we include the vector of parameters, μ, as a variable. Rewriting our

control system, equation III.2, for convenience, we have

$$\dot{x} = f(x,\mu) + g(x,\mu)\,u \tag{III.32}$$

Now we can use functional order form to write the expansions

$$\begin{aligned} f(x,\mu) &= f^{(1)}(x,\mu) + f^{(2)}(x,\mu) + f^{(3)}(x,\mu) + O^{(4+)} && \text{(III.33)} \\ g(x,\mu) &= g^{(0)}(x,\mu) + g^{(1)}(x,\mu) + g^{(2)}(x,\mu) + O^{(3+)} && \text{(III.34)} \end{aligned}$$

where we have used the fact that $f^{(0)}(x,\mu) = 0$ since the origin of coordinates has already been translated to a trimmed equilibrium point. Substituting these expansions into equation III.32 and grouping by term order yields

$$\begin{aligned} \dot{x} &= f^{(1)}(x,\mu) + g^{(0)}(x,\mu)\,u \\ &+ f^{(2)}(x,\mu) + g^{(1)}(x,\mu)\,u \\ &+ f^{(3)}(x,\mu) + g^{(2)}(x,\mu)\,u + O^{(4+)} \end{aligned} \tag{III.35}$$

which is the Taylor series expansion of our control system about the origin in functional order form, and where we have counted the control input u as order one. We desire the linear terms to be in vector/matrix form (we will leave the quadratic terms in functional order form for now), and since we are preparing the system for a linear similarity transformation in the next section, we need the coefficient matrix to be square. So, using the fact that $\dot{\mu} = 0$ (since μ is a constant) we can append equation III.35 and convert the linear terms to vector/matrix form to yield the block form

$$\begin{aligned} \begin{bmatrix} \dot{\mu} \\ \dot{x} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ L_{f_\mu} & L_{f_x} \end{bmatrix} \begin{bmatrix} \mu \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ G_0 \end{bmatrix} u \\ &+ \begin{bmatrix} 0 \\ f^{(2)}(x,\mu) \end{bmatrix} + \begin{bmatrix} 0 \\ g^{(1)}(x,\mu) \end{bmatrix} u \\ &+ \begin{bmatrix} 0 \\ f^{(3)}(x,\mu) \end{bmatrix} + \begin{bmatrix} 0 \\ g^{(2)}(x,\mu) \end{bmatrix} u + O^{(4+)} \end{aligned} \tag{III.36}$$

where we have used

$$f^{(1)}(x,\mu) = \begin{bmatrix} L_{f_\mu} & L_{f_x} \end{bmatrix} \begin{bmatrix} \mu \\ x \end{bmatrix} \tag{III.37}$$

and

$$g^{(0)}(x,\mu) = G_0 \tag{III.38}$$

Equation III.36 is what we will apply our linear similarity transformation and state feedback to in the next two sections to achieve our desired linear normal form.

C. LINEAR SIMILARITY TRANSFORMATION

As we will see in later chapters, the structure of the linear terms makes a tremendous difference to the simplification and transformation of the non-linear terms, so it is to our advantage to simplify the linear portion of the system as much as possible. Our first theorem shows that our system can be transformed into Jordan-Brunovsky canonical form through the application of a similarity transformation and linear state feedback. First, we define the Jordan-Brunovsky canonical form.

Definition C.1 (Jordan-Brunovsky Canonical Form) *A linear control system is said to be in Jordan-Brunovsky canonical form if it has the block vector/matrix form*

$$\begin{bmatrix} \dot{\mu} \\ \dot{z} \\ \dot{w} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ F_\mu & F_z & 0 & 0 \\ 0 & 0 & F_w & 0 \\ 0 & 0 & 0 & A \end{bmatrix} \begin{bmatrix} \mu \\ z \\ w \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ B \end{bmatrix} v \tag{III.39}$$

where $\mu \in R^r$ is a vector of parameters, $z \in R^q$ is a vector of linearly uncontrollable states having zero real-part eigenvalues, $w \in R^m$ is a vector of linearly uncontrollable states having non-zero real-part eigenvalues, $y \in R^p$ is a vector of linearly controllable states, and $v \in R^1$ is a single control input. The matrix $F_z \in R^{q\times q}$ is in block diagonal (Jordan) form, and all the eigenvalues of F_z have zero real parts. The matrix $F_\mu \in R^{q\times r}$ is in general non-square and has zero rows corresponding to the non-zero rows in the matrix F_z. The matrix $F_w \in R^{m\times m}$ is in block diagonal (Jordan) form, and all eigenvalues of the matrix F_w have non-zero real parts. The matrices

$A \in R^{p\times p}$ and $B \in R^{p\times 1}$ are in Brunovsky form, given by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & \cdots & & 0 \end{bmatrix} \tag{III.40}$$

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{III.41}$$

This definition allows for various degeneracies, that is, if certain of the vectors and corresponding matrices are missing, but otherwise the system is in Jordan-Brunovsky canonical form, then the system can be considered to be in the canonical form. For example, in certain systems of this form, the vector w (and by implication the matrix F_w) may not exist. Or, the vector z (and by implication the matrices F_μ and F_z) may not exist, etc. For the purposes of this dissertation, no distinction in terminology will be made between such degenerate cases and the complete Jordan-Brunovsky canonical form. Now we can state the Jordan-Brunovsky canonical form theorem.

Theorem C.2 (Jordan-Brunovsky Canonical Form) *Any linear system having the block matrix form*

$$\begin{bmatrix} \dot{\mu} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ L_{f_\mu} & L_{f_x} \end{bmatrix} \begin{bmatrix} \mu \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ G_0 \end{bmatrix} u \tag{III.42}$$

can be transformed into Jordan-Brunovsky canonical form through the application of a proper similarity transformation and state feedback. The required similarity transformation is given by

$$\begin{bmatrix} \mu \\ x \end{bmatrix} = T \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} \tag{III.43}$$

where the similarity transformation matrix, T, and its inverse, T^{-1}, are given by the block formulas

$$T = \begin{bmatrix} I & 0 \\ T_x T_\mu & T_x \end{bmatrix} \tag{III.44}$$

$$T^{-1} = \begin{bmatrix} I & 0 \\ -T_\mu & T_x^{-1} \end{bmatrix} \tag{III.45}$$

The sub-transformation matrix T_x is chosen so as to put L_{f_x} into block diagonal form

$$T_x^{-1} L_{f_x} T_x = \begin{bmatrix} F_z & 0 & 0 \\ 0 & F_w & 0 \\ 0 & 0 & \tilde{A} \end{bmatrix} \tag{III.46}$$

where F_z and F_w are in the appropriate block diagonal (Jordan) canonical form, and $\tilde{A}$ is in controllable canonical form. The sub-transformation matrix T_μ is chosen so as to eliminate as many rows as possible of the combination

$$T_x^{-1} L_{f_\mu} + T_x^{-1} L_{f_x} T_x T_\mu = \begin{bmatrix} F_\mu \\ 0 \\ \tilde{A}_\mu \end{bmatrix} \tag{III.47}$$

A subsequent theorem will show that all the rows of F_μ can be eliminated except for those corresponding to zero rows in F_z, and that $\tilde{A}_\mu$ can be eliminated entirely in most cases. (In those cases where $\tilde{A}_\mu$ cannot be eliminated entirely, only the bottom row will be left, which will be removed with state feedback.) The state feedback required is given by

$$u = \tilde{u} - \alpha^T \mu - a^T \tilde{y} \tag{III.48}$$

where α^T is the bottom row of $\tilde{A}_\mu$ (zero in most cases), and where a^T is the bottom row of $\tilde{A}$.

Proof. We wish to take our system

$$\begin{bmatrix} \dot{\mu} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ L_{f_\mu} & L_{f_x} \end{bmatrix} \begin{bmatrix} \mu \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ G_0 \end{bmatrix} u \tag{III.49}$$

and transform it into

$$\begin{bmatrix} \dot{\mu} \\ \dot{\tilde{z}} \\ \dot{\tilde{w}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ F_\mu & F_z & 0 & 0 \\ 0 & 0 & F_w & 0 \\ 0 & 0 & 0 & A \end{bmatrix} \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ B \end{bmatrix} \tilde{u} \tag{III.50}$$

We start by applying our linear transformation

$$\begin{bmatrix} \mu \\ x \end{bmatrix} = T \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} \tag{III.51}$$

with

$$T = \begin{bmatrix} I & 0 \\ T_x T_\mu & T_x \end{bmatrix} \tag{III.52}$$

$$T^{-1} = \begin{bmatrix} I & 0 \\ -T_\mu & T_x^{-1} \end{bmatrix} \tag{III.53}$$

to our original system. We get

$$\begin{bmatrix} \dot{\mu} \\ \dot{\tilde{z}} \\ \dot{\tilde{w}} \\ \dot{\tilde{y}} \end{bmatrix} = T^{-1} \begin{bmatrix} 0 & 0 \\ L_{f_\mu} & L_{f_x} \end{bmatrix} T \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} + T^{-1} \begin{bmatrix} 0 \\ G_0 \end{bmatrix} u \tag{III.54}$$

$$= \begin{bmatrix} I & 0 \\ -T_\mu & T_x^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ L_{f_\mu} & L_{f_x} \end{bmatrix} \begin{bmatrix} I & 0 \\ T_x T_\mu & T_x \end{bmatrix} \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix}$$

$$+ \begin{bmatrix} I & 0 \\ -T_\mu & T_x^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ G_0 \end{bmatrix} u$$

$$= \begin{bmatrix} 0 & 0 \\ T_x^{-1} L_{f_\mu} + T_x^{-1} L_{f_x} T_x T_\mu & T_x^{-1} L_{f_x} T_x \end{bmatrix} \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 0 \\ T_x^{-1} G_0 \end{bmatrix} u$$

Now, T_x is the similarity transformation which will separate out the controllable and uncontrollable modes for us, stacked in the right order, i.e.

$$T_x^{-1} L_{f_x} T_x = \begin{bmatrix} F_z & 0 & 0 \\ 0 & F_w & 0 \\ 0 & 0 & \tilde{A} \end{bmatrix} \tag{III.55}$$

$$T_x^{-1} G_0 = \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix} \tag{III.56}$$

We are also given that T_μ is a linear transformation which produces F_μ and $\tilde{A}_\mu$, eliminating as many of the entries as possible. (The next theorem will prove that this is possible, and give the structure of F_μ and $\tilde{A}_\mu$.)

$$T_x^{-1} L_{f_\mu} + T_x^{-1} L_{f_x} T_x T_\mu = \begin{bmatrix} F_\mu \\ 0 \\ \tilde{A}_\mu \end{bmatrix} \tag{III.57}$$

So, plugging in, we have our first intermediate result

$$\begin{bmatrix} \dot{\mu} \\ \dot{\tilde{z}} \\ \dot{\tilde{w}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ F_\mu & F_z & 0 & 0 \\ 0 & 0 & F_w & 0 \\ \tilde{A}_\mu & 0 & 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ B \end{bmatrix} u \tag{III.58}$$

Now apply the specified feedback to the system

$$u = \tilde{u} - \alpha^T \mu - a^T \tilde{y} \tag{III.59}$$

which gives

$$\begin{bmatrix} \dot{\mu} \\ \dot{\tilde{z}} \\ \dot{\tilde{w}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ F_\mu & F_z & 0 & 0 \\ 0 & 0 & F_w & 0 \\ \tilde{A}_\mu - B\alpha^T & 0 & 0 & \tilde{A} - Ba^T \end{bmatrix} \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ B \end{bmatrix} \tilde{u} \tag{III.60}$$

Since $\tilde{A}$ was previously defined as being in controllable canonical form, with a^T its bottom row, and $\tilde{A}_\mu$ was previously defined as being all zeros, except for the bottom row which was α^T, and since we know the values of B, α^T and a^T, we can calculate

$$\begin{aligned} \tilde{A}_\mu - B\alpha^T &= \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \alpha_1 & \cdots & \alpha_r \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \alpha_1 & \cdots & \alpha_r \end{bmatrix} \\ &= 0 \end{aligned} \tag{III.61}$$

and

$$\begin{aligned} \tilde{A} - Ba^T &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \\ a_1 & & \cdots & & a_p \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} a_1 & \cdots & a_r \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & \cdots & & 0 \end{bmatrix} = A \end{aligned} \tag{III.62}$$

which gives our desired result

$$\begin{bmatrix} \dot{\mu} \\ \dot{\tilde{z}} \\ \dot{\tilde{w}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ F_\mu & F_z & 0 & 0 \\ 0 & 0 & F_w & 0 \\ 0 & 0 & 0 & A \end{bmatrix} \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ B \end{bmatrix} \tilde{u} \tag{III.63}$$

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Our next theorem cleans up some results which were left hanging from the first theorem.

Theorem C.3 *The transformation matrix T_μ can be chosen in all cases such that all entries in the matrices F_μ and A_μ are zero, except for rows in F_μ corresponding to zero rows in F_z, and except for the last row in A_μ. In most cases, the transformation matrix T_μ can be chosen such that the matrix A_μ is entirely zero, which is entirely dependent on whether a single entry in the matrix A is non-zero.*

Proof. From the Jordan-Brunovsky canonical form theorem we have

$$T_x^{-1} L_{f_x} T_x = \begin{bmatrix} F_z & 0 & 0 \\ 0 & F_w & 0 \\ 0 & 0 & \tilde{A} \end{bmatrix} \tag{III.64}$$

and

$$T_x^{-1} G_0 = \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix} \tag{III.65}$$

which defined the transformation matrix T_x. We also have

$$T_x^{-1} L_{f_\mu} + T_x^{-1} L_{f_x} T_x T_\mu = \begin{bmatrix} F_\mu \\ 0 \\ \tilde{A}_\mu \end{bmatrix} \tag{III.66}$$

which now requires us to find T_μ. Breaking up the matrices into blocks grouped by $\tilde{y}$, $\tilde{w}$, and $\tilde{z}$ terms gives us

$$T_\mu = \begin{bmatrix} T_{\mu_z} \\ T_{\mu_w} \\ T_{\mu_y} \end{bmatrix} \tag{III.67}$$

$$T_x^{-1} L_{f_\mu} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \tag{III.68}$$

and substituting in yields

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} + \begin{bmatrix} F_z & 0 & 0 \\ 0 & F_w & 0 \\ 0 & 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} T_{\mu_z} \\ T_{\mu_w} \\ T_{\mu_y} \end{bmatrix} = \begin{bmatrix} F_\mu \\ 0 \\ \tilde{A}_\mu \end{bmatrix} \tag{III.69}$$

Since our equation is block diagonal it separates, allowing us to solve for T_{μ_z}, T_{μ_w} and T_{μ_y} independently. We get

$$F_\mu = 0 \tag{III.70}$$
$$T_{\mu_z} = -F_z^{-1}T_1 \tag{III.71}$$

if F_z^{-1} exists. We also get

$$T_{\mu_w} = -F_w^{-1}T_2 \tag{III.72}$$

since F_w^{-1} always exists. (By definition, all the eigenvalues of F_w have non-zero real parts. Since there are no possible zero eigenvalues, the matrix is invertible.) And, we get

$$\tilde{A}_\mu = 0 \tag{III.73}$$
$$T_{\mu_y} = -\tilde{A}^{-1}T_3 \tag{III.74}$$

if $\tilde{A}^{-1}$ exists. Now, look at the third block equation,

$$T_3 + \tilde{A}T_{\mu_y} = \tilde{A}_\mu \tag{III.75}$$

Looking at the individual rows of this equation, we have

$$\begin{bmatrix} T_{3_1} \\ \\ \vdots \\ \\ T_{3_p} \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ a_{p1} & & \cdots & & a_{pp} \end{bmatrix} \begin{bmatrix} T_{\mu_{y_1}} \\ \\ \vdots \\ \\ T_{\mu_{y_p}} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{\mu_1} \\ \\ \vdots \\ \\ \tilde{A}_{\mu_p} \end{bmatrix} \tag{III.76}$$

For $i = 1$ to $p-1$, the row equations are

$$\tilde{A}_{\mu_i} = T_{3_i} + T_{\mu_{y_{i+1}}} \tag{III.77}$$

so, if we pick $T_{\mu_{y_{i+1}}} = -T_{3_i}$, we get

$$\tilde{A}_{\mu_i} = 0 \tag{III.78}$$

For the bottom row, the equation is

$$\tilde{A}_{\mu_p} = T_{3_p} + \sum_{i=2}^{p} a_{pi} T_{\mu_{y_i}} + a_{p1} T_{\mu_{y_1}} \tag{III.79}$$

which we solve for $T_{\mu_{y_1}}$ if $a_{p1} \neq 0$, and get $\tilde{A}_{\mu_p} = 0$. If $a_{p1} = 0$ however, $T_{\mu_{y_1}}$ cannot affect the bottom row equation, and we solve for $\tilde{A}_{\mu_p}$

$$\tilde{A}_{\mu_p} = T_{3_p} + \sum_{i=2}^{p} a_{pi} T_{\mu_{y_i}} \tag{III.80}$$

So, all rows of $\tilde{A}_\mu$ can be forced to zero when $a_{p1} \neq 0$, and all rows of $\tilde{A}_\mu$ except for the bottom row can be forced to zero when $a_{p1} = 0$. Now, we look at the first block equation

$$T_1 + F_z T_{\mu_z} = F_\mu \tag{III.81}$$

Since the matrix F_z is block diagonal, this equation separates into sub-block equations as well. So, look at an individual block. There are three possible types of blocks. For a block consisting of a single zero eigenvalue, the rows of T_{μ_z} are multiplied by zero, and thus can have no effect on suppressing terms in F_μ. For a block consisting of a Jordan block of arbitrary dimension, all of the rows in the block of F_μ corresponding to the non-zero rows in the F_z Jordan block can be forced to zero. But, the bottom row of the Jordan block is all zeros, and cannot affect terms in F_μ. Finally, a block in F_z consisting of a pair of complex conjugates is invertible, and so can suppress all the corresponding rows in F_μ. So, any rows in F_z which are non-zero suppress the corresponding row in F_μ, and the only possible non-zero rows in F_μ correspond to zero rows in F_z. This proves the theorem. ◁

Now we need to apply the Jordan-Brunovsky canonical form theorem to the case of our control system, equation III.36, which we rewrite here for convenience

$$\begin{bmatrix} \dot{\mu} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ L_{f_\mu} & L_{f_x} \end{bmatrix} \begin{bmatrix} \mu \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ G_0 \end{bmatrix} u + \begin{bmatrix} 0 \\ f^{(2)}(x,\mu) \end{bmatrix} + \begin{bmatrix} 0 \\ g^{(1)}(x,\mu) \end{bmatrix} u \tag{III.82}$$

$$+\begin{bmatrix} 0 \\ f^{(3)}(x,\mu) \end{bmatrix} + \begin{bmatrix} 0 \\ g^{(2)}(x,\mu) \end{bmatrix} u \; + O^{(4+)}$$

The theorem says that we can find a linear transformation matrix, T, and a feedback control, $\tilde{u}$, such that the linear terms of our equation are put into Jordan-Brunovsky canonical form. But when we apply the theorem to the linear part, it also affects the non-linear part, and we get

$$\begin{bmatrix} \dot{\mu} \\ \dot{\tilde{z}} \\ \dot{\tilde{w}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ F_\mu & F_z & 0 & 0 \\ 0 & 0 & F_w & 0 \\ 0 & 0 & 0 & A \end{bmatrix} \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ B \end{bmatrix} \tilde{u} \tag{III.83}$$

$$+ \; T^{-1} \begin{bmatrix} 0 \\ f^{(2)}\left(T \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} \right) \end{bmatrix} + T^{-1} \begin{bmatrix} 0 \\ g^{(1)}\left(T \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} \right) \end{bmatrix} \left(\tilde{u} - \alpha^T \mu - a^T \tilde{y}\right)$$

$$+ \; T^{-1} \begin{bmatrix} 0 \\ f^{(3)}\left(T \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} \right) \end{bmatrix} + T^{-1} \begin{bmatrix} 0 \\ g^{(2)}\left(T \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} \right) \end{bmatrix} \left(\tilde{u} - \alpha^T \mu - a^T \tilde{y}\right)$$

$$+ \; O^{(4+)}$$

where we have used

$$(\mu, x) = \begin{bmatrix} \mu \\ x \end{bmatrix} = T \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} \tag{III.84}$$

and

$$u = \tilde{u} - \alpha^T \mu - a^T \tilde{y} \tag{III.85}$$

If we define

$$\tilde{f}^{(2)}(\mu,\tilde{z},\tilde{w},\tilde{y}) = T^{-1}\left(\begin{bmatrix} 0 \\ f^{(2)}\left(T\begin{bmatrix}\mu\\ \tilde{z}\\ \tilde{w}\\ \tilde{y}\end{bmatrix}\right)\end{bmatrix} - \begin{bmatrix} 0 \\ g^{(1)}\left(T\begin{bmatrix}\mu\\ \tilde{z}\\ \tilde{w}\\ \tilde{y}\end{bmatrix}\right)\end{bmatrix}\left(\alpha^T\mu + a^T\tilde{y}\right)\right) \tag{III.86}$$

$$\tilde{f}^{(3)}(\mu,\tilde{z},\tilde{w},\tilde{y}) = T^{-1}\left(\begin{bmatrix} 0 \\ f^{(3)}\left(T\begin{bmatrix}\mu\\ \tilde{z}\\ \tilde{w}\\ \tilde{y}\end{bmatrix}\right)\end{bmatrix} - \begin{bmatrix} 0 \\ g^{(2)}\left(T\begin{bmatrix}\mu\\ \tilde{z}\\ \tilde{w}\\ \tilde{y}\end{bmatrix}\right)\end{bmatrix}\left(\alpha^T\mu + a^T\tilde{y}\right)\right) \tag{III.87}$$

and

$$\tilde{g}^{(1)}(\mu,\tilde{z},\tilde{w},\tilde{y}) = T^{-1}\begin{bmatrix} 0 \\ g^{(1)}\left(T\begin{bmatrix}\mu\\ \tilde{z}\\ \tilde{w}\\ \tilde{y}\end{bmatrix}\right)\end{bmatrix} \tag{III.88}$$

$$\tilde{g}^{(2)}(\mu,\tilde{z},\tilde{w},\tilde{y}) = T^{-1}\begin{bmatrix} 0 \\ g^{(2)}\left(T\begin{bmatrix}\mu\\ \tilde{z}\\ \tilde{w}\\ \tilde{y}\end{bmatrix}\right)\end{bmatrix} \tag{III.89}$$

we can rewrite the final result more simply as

$$\begin{bmatrix}\dot{\mu}\\ \dot{\tilde{z}}\\ \dot{\tilde{w}}\\ \dot{\tilde{y}}\end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ F_\mu & F_z & 0 & 0 \\ 0 & 0 & F_w & 0 \\ 0 & 0 & 0 & A \end{bmatrix}\begin{bmatrix}\mu\\ \tilde{z}\\ \tilde{w}\\ \tilde{y}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ B\end{bmatrix}\tilde{u} \tag{III.90}$$

$$
\begin{aligned}
&+ \quad \tilde{f}^{(2)}(\mu, \tilde{z}, \tilde{w}, \tilde{y}) + \tilde{g}^{(1)}(\mu, \tilde{z}, \tilde{w}, \tilde{y})\,\tilde{u} \\
&+ \quad \tilde{f}^{(3)}(\mu, \tilde{z}, \tilde{w}, \tilde{y}) + \tilde{g}^{(2)}(\mu, \tilde{z}, \tilde{w}, \tilde{y})\,\tilde{u} + O^{(4+)}
\end{aligned}
$$

We say that an equation which has been put into the form of equation III.90 has been put into linear normal form.

We end this chapter with an example.

Example. [Linear Normal Form] Consider the rather complicated linear system

$$\dot{x}_1 = \frac{5}{2}\mu_1 - 1x_1 - 1x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4 + u \tag{III.91}$$

$$\dot{x}_2 = -\frac{1}{2}\mu - 1x_1 - 1x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_4 - u \tag{III.92}$$

$$\dot{x}_3 = -\frac{1}{2}\mu_1 + 1x_1 + 1x_2 + \frac{3}{2}x_3 - \frac{3}{2}x_4 + u \tag{III.93}$$

$$\dot{x}_4 = -\frac{3}{2}\mu_1 + 1x_1 + 1x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_4 - u \tag{III.94}$$

Appending the parameter μ_1 and putting the system into vector/matrix form gives

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{5}{2} & -1 & -1 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -1 & -1 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & 1 & \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 1 & 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} u \tag{III.95}$$

Inspection of this system reveals that it is in the appropriate form to apply the Jordan-Brunovsky canonical form theorem, with

$$L_{f_x} = \begin{bmatrix} -1 & -1 & \frac{1}{2} & -\frac{1}{2} \\ -1 & -1 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 1 & \frac{3}{2} & -\frac{3}{2} \\ 1 & 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{III.96}$$

$$L_{f_\mu} = \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix} \tag{III.97}$$

and

$$G_0 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \tag{III.98}$$

From the theorem, we have to generate two transformation matrices, T_x and T_μ, such that the block form equations

$$T_x^{-1} L_{f_x} T_x = \begin{bmatrix} F_z & 0 & 0 \\ 0 & F_w & 0 \\ 0 & 0 & \tilde{A} \end{bmatrix} \tag{III.99}$$

and

$$T_x^{-1} G_0 = \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix} \tag{III.100}$$

define the transformation matrix T_x, and the block form equation

$$T_x^{-1} L_{f_\mu} + T_x^{-1} L_{f_x} T_x T_\mu = \begin{bmatrix} F_\mu \\ 0 \\ \tilde{A}_\mu \end{bmatrix} \tag{III.101}$$

defines the transformation matrix T_μ. We find T_x and T_x^{-1} to be

$$T_x = \begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \tag{III.102}$$

$$T_x^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \tag{III.103}$$

which gives

$$T_x^{-1} L_{f_x} T_x = \begin{bmatrix} F_z & 0 & 0 \\ 0 & F_w & 0 \\ 0 & 0 & \tilde{A} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \text{(III.104)}$$

This gives

$$F_z = [0] \qquad \text{(III.105)}$$

$$F_w = [-2] \qquad \text{(III.106)}$$

$$\tilde{A} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \qquad \text{(III.107)}$$

We find T_μ to be

$$T_\mu = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \qquad \text{(III.108)}$$

which gives

$$T_x^{-1} L_{f_\mu} + T_x^{-1} L_{f_x} T_x T_\mu = \begin{bmatrix} F_\mu \\ 0 \\ \tilde{A}_\mu \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix} \qquad \text{(III.109)}$$

So, after the similarity transformation our system is

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{\tilde{z}}_1 \\ \dot{\tilde{w}}_1 \\ \dot{\tilde{y}}_1 \\ \dot{\tilde{y}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \tilde{z}_1 \\ \tilde{w}_1 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \qquad \text{(III.110)}$$

Now apply linear state feedback of the form

$$\tilde{u} = 3\mu_1 + 2\tilde{y}_2 + u \qquad \text{(III.111)}$$

which gives our system in Jordan-Brunovsky canonical form as

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{z}_1 \\ \dot{w}_1 \\ \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \tilde{z}_1 \\ \tilde{w}_1 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tilde{u} \tag{III.112}$$

with the canonical form submatrices given by

$$F_\mu = [1] \tag{III.113}$$

$$F_z = [0] \tag{III.114}$$

$$F_w = [-2] \tag{III.115}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{III.116}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{III.117}$$

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IV. CONTROL OF LINEARLY CONTROLLABLE, LINEARLY STABILIZABLE AND LINEARLY UNSTABLE BIFURCATIONS

A. ROADMAP: THE BIG PICTURE

1. Results of Previous Chapters

In Chapters II and III we showed that any affine system

$$\dot{\breve{x}} = \breve{f}\left(\breve{x}, \breve{\mu}\right) + \breve{g}\left(\breve{x}, \breve{\mu}\right)\breve{u} \tag{IV.1}$$

can be put into linear normal form

$$\begin{bmatrix} \dot{\mu} \\ \dot{\tilde{z}} \\ \dot{\tilde{w}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ F_\mu & F_z & 0 & 0 \\ 0 & 0 & F_w & 0 \\ 0 & 0 & 0 & A \end{bmatrix} \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ B \end{bmatrix} \tilde{u} + O^{(2+)} \tag{IV.2}$$

where $\mu \in R^r$ is the vector of parameters, $\tilde{z} \in R^q$ is the vector of linearly uncontrollable states having zero real-part eigenvalues, $\tilde{w} \in R^m$ is the vector of linearly uncontrollable states having non-zero real-part eigenvalues, $\tilde{y} \in R^p$ is the vector of linearly controllable states, and $\tilde{u} \in R^1$ is a single control input. Also, the matrix F_z is in block diagonal (Jordan) form, and all the eigenvalues of F_z have zero real parts, the matrix F_μ has zero rows corresponding to non-zero rows in F_z, the matrix F_w is in block diagonal (Jordan) form, and all eigenvalues of the matrix F_w have negative real parts, and the matrices A and B are in Brunovsky form of appropriate dimension. (In certain systems of this form, the vector $\tilde{w}$ (and by implication the matrix F_w) and/or the vector $\tilde{z}$ (and by implication the matrices F_μ and F_z) may not exist. That is, all the linearly uncontrollable states may have eigenvalues with only zero real parts, or only non-zero real parts, or there may be no linearly uncontrollable states at all. In these cases, deleting all reference to the states $\tilde{w}$ and/or $\tilde{z}$ as appropriate in the general equations gives the right answer as we will see.) We can also pick the control $\tilde{u} = \tilde{u}\left(\mu, \tilde{z}, \tilde{w}, \tilde{y}\right)$ as state feedback.

2. Purpose of this Chapter

In this chapter we apply linear control theory to stabilize the linearly controllable states $\tilde{y}$ found in the linear normal form of our system. We show that the question of whether or not a trimmed system of Chapter II experiences a bifurcation is an independent question from that of whether such a system is linearly controllable. We will see that there are four possible cases for a system which does experience a bifurcation at a trimmed equilibrium point: the system can turn out to be linearly controllable or stabilizable; the system can turn out to be linearly unstable; the system can experience the full bifurcation in the linearly uncontrollable states; or the system can experience a reduced dimension bifurcation in the linearly uncontrollable states. This chapter considers the linearly controllable/stabilizable case and the linearly unstable case. Chapters V, VI and VII consider how to stabilize bifurcations occurring in the linearly uncontrollable states.

B. THE GENERAL METHOD FOR STABILIZING SYSTEMS WITH BIFURCATIONS

The general method we will employ in the stabilization of all of the bifurcations we will consider in this chapter and in Chapter VII consists of the following steps:

1. Determine if a bifurcation occurs in the system of interest using the method of Chapter II. If so, trim the system to, and translate the origin of coordinates to, the equilibrium point of interest at the point of bifurcation.
2. Determine the linear properties of the trimmed system by transforming the system into linear normal form using the method of Chapter III. There are four possible cases, and we will consider the first three in this chapter, and the fourth in Chapter VII:
    - The system is linearly controllable. That is, all states (except for the appended vector of parameters) are linearly controllable. This case is considered in this chapter.
    - The system is linearly stabilizable. That is, all linearly uncontrollable states (except for the appended vector of parameters) have eigenvalues with negative real parts and are exponentially stable. This case is considered in this chapter.

- The system is linearly unstable. That is, at least one linearly uncontrollable state has an eigenvalue with a positive real part and is exponentially unstable. This case is considered in this chapter.
- The system is linearly unstabilizable, but not linearly unstable. That is, all linearly uncontrollable states have eigenvalues with either negative or zero real parts, but there are no eigenvalues with positive real parts. This case requires consideration of the non-linear terms to determine stability, and will be considered in Chapter VII.

3. Apply linear control techniques to stabilize the linearly controllable states. Any stabilizing linear control method (pole placement, linear quadratic regulator, robust control, etc.) is acceptable.

4. If the system is linearly unstable, then stabilization is not possible using the methods in this dissertation.

5. If the system is linearly unstabilizable, but not linearly unstable, then the methods of Chapters V through VII must be used to stabilize the system using the non-linear terms.

6. Transform the stabilizing feedback into the original system by reversing all the translations and transformations used to put the system into normal form. The process for reversing the linear normal form transformations is detailed at the end of this chapter, and the process for reversing the quadratic normal form transformations (including the linear normal form transformations as a subset) is detailed at the end of Chapter V.

In this chapter we will consider the control or stabilization of a dynamic control system which experiences a bifurcation when the system is linearly controllable, linearly stabilizable, or linearly unstable.

C. BIFURCATIONS AND LINEAR CONTROLLABILITY OR STABILIZABILITY

In Chapter II we found that a system may undergo a bifurcation when it is trimmed to an equilibrium point, without ever considering whether the system was linearly controllable/stabilizable or not. In fact, the two questions, "Does a system experience a bifurcation?" and, "Is the system linearly controllable/stabilizable/unstable?"

are independent questions, as we see if we can see if we consider the following examples.

Example. [Linearly Controllable Bifurcation] Consider the simple dynamic system

$$\dot{\breve{x}}_1 = \breve{\mu}_1 + \breve{x}_1^2 + \breve{x}_2 \tag{IV.3}$$

$$\dot{\breve{x}}_2 = \breve{u} \tag{IV.4}$$

Using the methods of Chapter II, we find that this system can only be trimmed at an equilibrium point when

$$\breve{u}^* = 0 \tag{IV.5}$$

and that we have two free variables, $\breve{\mu}_1$ and $\breve{x}_2^*$, which determine our equilibrium set as

$$\breve{x}_1^* = \pm\sqrt{-\breve{\mu}_1 - \breve{x}_2^*} \tag{IV.6}$$

A saddle-node bifurcation occurs in this system, with the bifurcation point at

$$\breve{\mu}_1^* = -\breve{x}_2^* \tag{IV.7}$$

The equilibrium point at the point of bifurcation is

$$\breve{x}_2^* = arbitrary \tag{IV.8}$$

$$\breve{x}_1^* = 0 \tag{IV.9}$$

$$\breve{\mu}_1^* = -\breve{x}_2^* \tag{IV.10}$$

Translating the origin of our coordinate system to the equilibrium point at the point of bifurcation yields an equivalent trimmed system

$$\dot{x}_1 = \mu_1 + x_1^2 + x_2 \tag{IV.11}$$

$$\dot{x}_2 = u \tag{IV.12}$$

which, in the absence of control (that is for x_2 held constant — no feedback applied) still has a saddle-node bifurcation in the x_1 dynamics. Using the methods of Chapter

III, we can expand our system in a Taylor series, which yields

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ x_1^2 \\ 0 \end{bmatrix} \tag{IV.13}$$

then apply a coordinate change of the form

$$x_1 = \tilde{y}_1 \tag{IV.14}$$

$$x_2 = -\mu_1 + \tilde{y}_2 \tag{IV.15}$$

which yields a system in linear normal form

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{\tilde{y}}_1 \\ \dot{\tilde{y}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tilde{u} + \begin{bmatrix} 0 \\ \tilde{y}_1^2 \\ 0 \end{bmatrix} \tag{IV.16}$$

This system is linearly controllable! We can pick any stabilizing state feedback gains

$$K_y = \begin{bmatrix} K_{y_1} \\ K_{y_2} \end{bmatrix} \tag{IV.17}$$

such that the closed loop matrix

$$A + BK_y^T = \begin{bmatrix} 0 & 1 \\ K_{y_1} & K_{y_2} \end{bmatrix} \tag{IV.18}$$

is stable, which gives us our control law

$$\tilde{u} = K_{y_1}\tilde{y}_1 + K_{y_2}\tilde{y}_2 \tag{IV.19}$$

Reversing the coordinate transformations and translations to get back to our original system gives

$$\breve{u} = K_{y_1}\breve{x}_1 + K_{y_2}\left(\breve{x}_2 + \breve{\mu}_1\right) \tag{IV.20}$$

which is the control law which stabilizes our system at the equilibrium point $\breve{x}_1^* = 0$, $\breve{x}_2^* = arbitrary$. It should not be surprising that we were required to feedback the parameter $\breve{\mu}_1$ in order to stabilize our system. ◁

Example. [Linearly Uncontrollable Bifurcation] Now consider the simple dynamic system, slightly changed from the previous example,

$$\dot{\breve{x}}_1 = \breve{\mu}_1 + \breve{x}_1^2 + \breve{x}_2^2 \tag{IV.21}$$

$$\dot{\breve{x}}_2 = \breve{u} \tag{IV.22}$$

Using the methods of Chapter II, we again find that this system can only be trimmed at an equilibrium point when

$$\breve{u}^* = 0 \tag{IV.23}$$

and that we have two free variables, $\breve{\mu}_1$ and $\breve{x}_2^*$, which determine our equilibrium set as

$$\breve{x}_1^* = \pm\sqrt{-\breve{\mu}_1 - \breve{x}_2^{*2}} \tag{IV.24}$$

A saddle-node bifurcation occurs in this system, with the bifurcation point at

$$\mu_1^* = -x_2^{*2} \tag{IV.25}$$

The equilibrium point at the point of bifurcation is

$$\breve{x}_2^* = arbitrary \tag{IV.26}$$

$$\breve{x}_1^* = 0 \tag{IV.27}$$

$$\breve{\mu}_1^* = -\breve{x}_2^{*2} \tag{IV.28}$$

Translating the origin of our coordinate system to the equilibrium point at the point of bifurcation yields an equivalent trimmed system

$$\dot{x}_1 = \mu_1 + x_1^2 + 2\breve{x}_2^* x_2 + x_2^2 \tag{IV.29}$$

$$\dot{x}_2 = u \tag{IV.30}$$

which, in the absence of control (that is, for x_2 held constant — no feedback applied) has a saddle-node bifurcation in the x_1 dynamics. Using the methods of Chapter III,

we can expand our system in a Taylor series, which yields

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2\breve{x}_2^* \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ x_1^2 + x_1^2 \\ 0 \end{bmatrix} \tag{IV.31}$$

then apply a coordinate change of the form

$$x_1 = 2\breve{x}_2^*\tilde{y}_1 \tag{IV.32}$$

$$x_2 = -\left(\frac{1}{2\breve{x}_2^*}\right)\mu_1 + \tilde{y}_2 \tag{IV.33}$$

(for $\breve{x}_2^* \neq 0$), which yields a system in linear normal form

$$\begin{bmatrix} \mu_1 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tilde{u} + \begin{bmatrix} 0 \\ (2\breve{x}_2^*)^2 \tilde{y}_1^2 + \left(\frac{1}{2\breve{x}_2^*}\right)^2 \mu_1^2 - \left(\frac{1}{\breve{x}_2^*}\right)\mu_1\tilde{y}_2 + \tilde{y}_2^2 \\ 0 \end{bmatrix} \tag{IV.34}$$

Once again, this system is linearly controllable. We can pick stabilizing state feedback gains

$$K_y = \begin{bmatrix} K_{y_1} \\ K_{y_2} \end{bmatrix} \tag{IV.35}$$

such that the closed loop matrix

$$A + BK_y^T = \begin{bmatrix} 0 & 1 \\ K_{y_1} & K_{y_2} \end{bmatrix} \tag{IV.36}$$

is stable, which gives us our control law

$$\tilde{u} = K_{y_1}\tilde{y}_1 + K_{y_2}\tilde{y}_2 \tag{IV.37}$$

Reversing the coordinate transformations and translations to get back to our original system gives

$$\breve{u} = \left(\frac{K_{y_1}}{2\breve{x}_2^*}\right)\breve{x}_1 + K_{y_2}\left(\breve{x}_2 + \left(\frac{1}{2\breve{x}_2^*}\right)\breve{\mu}_1 - \frac{1}{2}\breve{x}_2^*\right) \tag{IV.38}$$

which is the control law which stabilizes our system at the equilibrium point $\breve{x}_1^* = 0$, $\breve{x}_2^* = arbitrary$. However, there is one small fly in the ointment: if our desired equilibrium point is $\breve{x}_2^* = 0$, then the above analysis breaks down. For $\breve{x}_2^* = 0$, our system in linear normal form is

$$\begin{bmatrix} \mu_1 \\ \tilde{z}_1 \\ \tilde{y}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \tilde{z}_1 \\ \tilde{y}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tilde{u} + \begin{bmatrix} 0 \\ \tilde{z}_1^2 + \tilde{y}_1^2 \\ 0 \end{bmatrix} \tag{IV.39}$$

where we used the obvious linear coordinate transformation

$$x_1 = \tilde{z}_1 \tag{IV.40}$$

$$x_2 = \tilde{y}_1 \tag{IV.41}$$

In this case, the saddle-node bifurcation occurring in the state $\tilde{x}_1$ is linearly uncontrollable, and the bifurcation is linearly unstabilizable without being linearly unstable. We will consider how to handle bifurcations of this form in Chapters V, VI and VII.

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D. LINEARLY CONTROLLABLE BIFURCATIONS

A bifurcation is linearly controllable if, after the system is transformed into linear normal form, the system has the form

$$\begin{bmatrix} \dot{\mu} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \mu \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} \tilde{u} + O^{(2+)} \tag{IV.42}$$

where the matrices A and B are in Brunovsky form of appropriate dimension. The system can be stabilized by choosing state feedback gains

$$K_y = \begin{bmatrix} K_{y_1} \\ \vdots \\ K_{y_p} \end{bmatrix} \tag{IV.43}$$

such that the real part of the eigenvalues of the closed loop matrix $A + BK_y^T$ are negative. The control law is given by

$$\tilde{u} = K_y^T \tilde{y} \tag{IV.44}$$

which, after reversing the transformations and translations needed to achieve linear normal form, gives the control law in the original system in block form as

$$\breve{u} = \breve{u}^* + \left(K_y^T - a^T\right) T_x^{-1} \left(\breve{x} - \breve{x}^*\right) - \left(\left(K_y^T - a^T\right) T_\mu + \alpha^T\right) \left(\breve{\mu} - \breve{\mu}^*\right) \tag{IV.45}$$

where the trim control $\breve{u}^*$, equilibrium point $\breve{x}^*$, and bifurcation point $\breve{\mu}^*$ were found using the method of Chapter II, and the transformation matrices T_x^{-1} and T_μ and the bottom row vectors a^T and α^T were found using the method of Chapter III.

E. LINEARLY STABILIZABLE BIFURCATIONS

A bifurcation is linearly stabilizable if, after the system is transformed into linear normal form, the system has the form

$$\begin{bmatrix} \dot{\mu} \\ \dot{\tilde{w}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & F_w & 0 \\ 0 & 0 & A \end{bmatrix} \begin{bmatrix} \mu \\ \tilde{w} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix} \tilde{u} + O^{(2+)} \tag{IV.46}$$

where the matrix F_w is in block diagonal (Jordan) form, and all eigenvalues of the matrix F_w have negative real parts, and the matrices A and B are in Brunovsky form of appropriate dimension. The system can again be stabilized by choosing state feedback gains

$$K_y = \begin{bmatrix} K_{y_1} \\ \vdots \\ K_{y_p} \end{bmatrix} \tag{IV.47}$$

such that the real part of the eigenvalues of the closed loop matrix $A + BK_y^T$ are negative. The control law is given by

$$\tilde{u} = K_y^T y \tag{IV.48}$$

which, after reversing the transformations and translations needed to achieve linear normal form, gives the control law in the original system in block form as

$$\breve{u} = \breve{u}^* + \left[\begin{array}{cc} 0 & \left(K_y^T - a^T\right) \end{array}\right] T_x^{-1} \left(\breve{x} - \breve{x}^*\right) - \left(\left[\begin{array}{cc} 0 & \left(K_y^T - a^T\right) \end{array}\right] T_\mu + \alpha^T\right) \left(\breve{\mu} - \breve{\mu}^*\right) \tag{IV.49}$$

where the trim control $\breve{u}^*$, equilibrium point $\breve{x}^*$, and bifurcation point $\breve{\mu}^*$ were found using the method of Chapter II, and the transformation matrices T_x^{-1} and T_μ and the bottom row vectors a^T and α^T were found using the method of Chapter III.

F. LINEARLY UNSTABLE BIFURCATIONS

A bifurcation is linearly unstable if, after the system is transformed into linear normal form, the system has linearly uncontrollable states with eigenvalues having positive real parts. In this situation, there is no method presented in this dissertation which can stabilize the system.

G. REVERSING THE TRANSFORMATIONS

In Chapter II we began with a system of the form

$$\dot{\breve{x}} = \breve{f}\left(\breve{x}, \breve{\mu}\right) + \breve{g}\left(\breve{x}, \breve{\mu}\right) \breve{u} \tag{IV.50}$$

and used translation, transformation and feedback to produce a system in the linear normal form of equation IV.2. In this form, the control law which stabilizes the system is simple, as it is just a vector of state feedback gains for the linearly controllable states, y. However, in the original system, the control law may be substantially more complicated. In this section we will develop a systematic formula which relates the simple control law developed from the linear normal form to the actual control law which must be applied to the original system. We begin by developing the most complete linear state feedback control law possible in the linear normal form system. We use linear state feedback gains multiplied by every possible state to get

$$\tilde{u} = K_\mu^T \mu + K_z^T \tilde{z} + K_w^T \tilde{w} + K_y^T \tilde{y} \tag{IV.51}$$

where we note in passing that the gain vectors K_μ^T, K_z^T and K_w^T have no effect on the stability of the closed loop linear system, which is determined solely by the gain vector K_y^T. Now, we want to unfold the control term and the states to get back to our original system, and we start with the control term $\tilde{u}$. From Chapter III we have the fact that $\tilde{u}$ was arrived at by feedback after a coordinate transformation. Repeating the equation for convenience, we have

$$\tilde{u} = u + \alpha^T \mu + a^T \tilde{y} \tag{IV.52}$$

which we can combine with equation IV.51 to get

$$u = \begin{bmatrix} \left(K_\mu^T - \alpha^T\right) & K_z^T & K_w^T & \left(K_y^T - a^T\right) \end{bmatrix} \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} \tag{IV.53}$$

Recalling the coordinate transformation in block form

$$\begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -T_\mu & T_x^{-1} \end{bmatrix} \begin{bmatrix} \mu \\ x \end{bmatrix} \tag{IV.54}$$

we can plug in to get

$$\begin{aligned} u \;=\; & \left(K_\mu^T - \alpha^T - \begin{bmatrix} K_z^T & K_w^T & \left(K_y^T - a^T\right) \end{bmatrix} T_\mu\right)\mu \\ & + \begin{bmatrix} K_z^T & K_w^T & \left(K_y^T - a^T\right) \end{bmatrix} T_x^{-1} x \end{aligned} \tag{IV.55}$$

Finally, recalling the translation of the origin and the trim control input

$$x = \breve{x} - \breve{x}^* \tag{IV.56}$$

$$\mu = \breve{\mu} - \breve{\mu}^* \tag{IV.57}$$

$$u = \breve{u} - \breve{u}^* \tag{IV.58}$$

we get

$$\begin{aligned} \breve{u} &= \breve{u}^* + \left(K_\mu^T - \alpha^T - \left[\begin{array}{ccc} K_z^T & K_w^T & \left(K_y^T - a^T \right) \end{array} \right] T_\mu \right) (\breve{\mu} - \breve{\mu}^*) \qquad \text{(IV.59)} \\ &\quad + \left[\begin{array}{ccc} K_z^T & K_w^T & \left(K_y^T - a^T \right) \end{array} \right] T_x^{-1} (\breve{x} - \breve{x}^*) \end{aligned}$$

which is our general linear control law in the original system.

In Chapters V, VI and VII we proceed on to those cases of real interest, namely those cases in which the bifurcation is linearly unstabilizable but not linearly unstable, that is, the bifurcation occurs in linearly uncontrollable states having eigenvalues with zero real parts, and there are no linearly uncontrollable states with eigenvalues having positive real parts.

V. QUADRATIC NORMAL FORM

A. ROADMAP: THE BIG PICTURE

1. Results of Previous Chapters

In Chapters II and III we showed that any affine control system

$$\dot{\breve{x}} = \breve{f}(\breve{x}, \breve{\mu}) + \breve{g}(\breve{x}, \breve{\mu})\,\breve{u} \tag{V.1}$$

could be put into linear normal form

$$\begin{aligned}
\begin{bmatrix} \dot{\mu} \\ \dot{\tilde{z}} \\ \dot{\tilde{w}} \\ \dot{\tilde{y}} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ F_\mu & F_z & 0 & 0 \\ 0 & 0 & F_w & 0 \\ 0 & 0 & 0 & A \end{bmatrix} \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ B \end{bmatrix} \tilde{u} \\
&+ \tilde{f}^{(2)}(\mu, \tilde{z}, \tilde{w}, \tilde{y}) + \tilde{g}^{(1)}(\mu, \tilde{z}, \tilde{w}, \tilde{y})\,\tilde{u} \\
&+ \tilde{f}^{(3)}(\mu, \tilde{z}, \tilde{w}, \tilde{y}) + \tilde{g}^{(2)}(\mu, \tilde{z}, \tilde{w}, \tilde{y})\,\tilde{u} + O^{(4+)}
\end{aligned} \tag{V.2}$$

through a translation of the origin of coordinates to an equilibrium point at the point of bifurcation, followed by a linear coordinate transformation and an application of linear state feedback. In Chapter IV we showed that the linearly controllable states y could be stabilized by the proper application of linear state feedback, and that if the system in linear normal form contained only linearly controllable states y and linearly uncontrollable but stable states w, that the bifurcation detected in Chapter II could be controlled or stabilized with linear state feedback. We also stated that nothing in this dissertation could stabilize the unstable case, when one or several of the linearly uncontrollable states w had eigenvalues with positive real parts.

2. Purpose of this Chapter

In this chapter we consider how to simplify the quadratic order terms in equation V.2. Our goal is to perform quadratic coordinate transformations and apply quadratic state feedback to transform equation V.2 into quadratic normal form, given

by

$$\dot{\mu} = 0 \tag{V.3}$$

$$\begin{aligned}
\dot{z} &= F_\mu \mu + F_z z \qquad\qquad (V.4)\\
&+ Q_{z_{p_1}} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + Q_{z_{m_1}} \begin{bmatrix} \mu y_1 \\ z y_1 \end{bmatrix} + Q_{z_c} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \\
&+ Q_{z_{p_2}} \begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix} + Q_{z_{p_3}} \begin{bmatrix} w^{(2)} \end{bmatrix} + Q_{z_{m_2}} \begin{bmatrix} w y_1 \end{bmatrix} \\
&+ \tilde{f}_z^{(3)}(\mu, z, w, y) + \tilde{g}_z^{(2)}(\mu, z, w, y)\, v + O^{(4+)} \qquad\qquad (V.5)
\end{aligned}$$

$$\begin{aligned}
\dot{w} &= F_w w \qquad\qquad (V.6)\\
&+ Q_{w_{p_1}} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + Q_{w_{m_1}} \begin{bmatrix} \mu y_1 \\ z y_1 \end{bmatrix} + Q_{w_c} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \\
&+ Q_{w_{p_2}} \begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix} + Q_{w_{p_3}} \begin{bmatrix} w^{(2)} \end{bmatrix} + Q_{w_{m_2}} \begin{bmatrix} w y_1 \end{bmatrix} + O^{(3+)}
\end{aligned}$$

$$\dot{y} = Ay + Bv + Q_{y_c} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} + O^{(3+)} \tag{V.7}$$

where $\mu \in R^r$ is the vector of parameters, $z \in R^q$ is the vector of linearly uncontrollable states having zero real-part eigenvalues, $w \in R^m$ is the vector of linearly uncontrollable states having negative real-part eigenvalues, $y \in R^p$ is the vector of linearly controllable states, $v \in R^1$ is a single control input, and the matrices F_μ, F_z, F_w, A and B are in the appropriate Jordan-Brunovsky canonical form. Subsequent

chapters will consider how to use this quadratic normal form to impose non-linear stability on linearly unstabilizable bifurcations.

B. THE QUADRATIC COORDINATE TRANSFORMATION AND QUADRATIC STATE FEEDBACK

Equation V.2 is the linear normal form of our control system. However, the appended state vector is too complicated to facilitate a clear understanding of the quadratic coordinate transformation process. So, we introduce a new variable name for the appended state vector, $\tilde{\chi} \in R^{\nu}$, and new notation for the block linear coefficient matrices, i.e.

$$\tilde{\chi} = \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} \tag{V.8}$$

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ F_\mu & F_z & 0 & 0 \\ 0 & 0 & F_w & 0 \\ 0 & 0 & 0 & A \end{bmatrix} . \tag{V.9}$$

$$G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ B \end{bmatrix} \tag{V.10}$$

This allows us to write the linear normal form of our control system, equation V.2, as

$$\begin{aligned} \dot{\tilde{\chi}} &= F\tilde{\chi} + G\tilde{u} + \tilde{f}^{(2)}(\tilde{\chi}) + \tilde{g}^{(1)}(\tilde{\chi})\,\tilde{u} \\ &+ \tilde{f}^{(3)}(\tilde{\chi}) + \tilde{g}^{(2)}(\tilde{\chi})\,\tilde{u} + O^{(4+)} \end{aligned} \tag{V.11}$$

Now, we want to perform a quadratic coordinate transformation to simplify our quadratic terms, but we don't want the transformation to alter the form of our linear

terms. An appropriate quadratic transformation is

$$\tilde{\chi} = \chi + h^{(2)}(\chi) \tag{V.12}$$

where $h^{(2)}(\chi)$ is an as-yet unknown quadratic function of χ, and $\chi \in R^{\nu}$ is our transformed state vector. As desired, this transformation only alters the quadratic and higher order terms. Now we will apply this transformation to our system.

1. The Homological and Constraint Equations

To arrive at our transformed dynamic system, we calculate $\dot{\chi}$ two ways and equate them. The first way is to plug our transformed state vector into our linear normal form, that is, plug equation V.12 into equation V.11, which gives

$$\begin{aligned} \dot{\tilde{\chi}} &= F\chi + G\tilde{u} + Fh^{(2)}(\chi) + \tilde{f}^{(2)}(\chi) + \tilde{g}^{(1)}(\chi)\,\tilde{u} \\ &+ \bar{f}^{(3)}(\chi) + \bar{g}^{(2)}(\chi)\,\tilde{u} + O^{(4+)} \end{aligned} \tag{V.13}$$

where we have defined

$$\bar{f}^{(3)}(\chi) + O^{(4+)} = \tilde{f}^{(3)}(\chi) + \left(\tilde{f}^{(2)}\left(\chi + h^{(2)}(\chi)\right) - \tilde{f}^{(2)}(\chi)\right) \tag{V.14}$$

$$\bar{g}^{(2)}(\chi) = \tilde{g}^{(2)}(\chi) + \tilde{g}^{(1)}\left(h^{(2)}(\chi)\right) \tag{V.15}$$

and used

$$\tilde{f}^{(3)}\left(\chi + h^{(2)}(\chi)\right) = \tilde{f}^{(3)}(\chi) + O^{(4+)} \tag{V.16}$$

$$\tilde{g}^{(2)}\left(\chi + h^{(2)}(\chi)\right) = \tilde{g}^{(2)}(\chi) + O^{(3+)} \tag{V.17}$$

The second way is to differentiate our quadratic coordinate transformation, which gives

$$\begin{aligned} \dot{\tilde{\chi}} &= \dot{\chi} + \frac{d}{dt}h^{(2)}(\chi) \\ &= \dot{\chi} + \frac{\partial}{\partial\chi}h^{(2)}(\chi)\,\dot{\chi} \\ &= \left(I + \frac{\partial}{\partial\chi}h^{(2)}(\chi)\right)\dot{\chi} \end{aligned} \tag{V.18}$$

where the function $\frac{\partial}{\partial\chi}h^{(2)}(\chi)$ is a square matrix-valued function of χ, that is, a square matrix whose entries are functions of χ. (In this case they are linear functions of χ.) Setting equation V.13 equal to equation V.18 gives

$$\begin{aligned}\dot{\chi} &= \left(I + \frac{\partial}{\partial\chi}h^{(2)}(\chi)\right)^{-1}\left(F\chi + G\tilde{u} + Fh^{(2)}(\chi) + \tilde{f}^{(2)}(\chi) + \tilde{g}^{(1)}(\chi)\,\tilde{u}\right) \\ &+ \left(I + \frac{\partial}{\partial\chi}h^{(2)}(\chi)\right)^{-1}\left(\tilde{f}^{(3)}(\chi) + \tilde{g}^{(2)}(\chi)\,\tilde{u} + O^{(4+)}\right)\end{aligned} \qquad \text{(V.19)}$$

Now we need to know how to invert a square matrix-valued function of χ. We show this in a lemma.

Lemma B.1 (Inverse of a Matrix Function) *A square matrix-valued function* $\Phi(\chi)$*, of the form*

$$\Phi(\chi) = I + \Phi^{(1)}(\chi) \qquad \text{(V.20)}$$

has an inverse $\Theta(\chi) \equiv \Phi^{-1}(\chi)$ *in a vicinity of* $\chi = 0$*, which is a square matrix-valued function having the Taylor series expansion*

$$\Theta(\chi) = \Theta^{(0)}(\chi) + \Theta^{(1)}(\chi) + \Theta^{(2)}(\chi) + \Theta^{(3)}(\chi) + O^{(4+)} \qquad \text{(V.21)}$$

where the terms for the Taylor series expansion for $\Theta(\chi)$ *are given by*

$$\begin{aligned}\Theta^{(0)}(\chi) &= I && \text{(V.22)}\\ \Theta^{(1)}(\chi) &= -\Phi^{(1)}(\chi) && \text{(V.23)}\\ \Theta^{(2)}(\chi) &= \left(-\Phi^{(1)}(\chi)\right)^2 && \text{(V.24)}\\ \Theta^{(3)}(\chi) &= \left(-\Phi^{(1)}(\chi)\right)^3 && \text{(V.25)}\\ &\vdots && \text{(V.26)}\end{aligned}$$

Proof. This lemma is a well-known result, which we demonstrate briefly here. Since $\Theta(\chi) \equiv \Phi^{-1}(\chi)$, multiplying $\Theta(\chi)$ by $\Phi(\chi)$ should yield the identity matrix. Multiplying term by term and matching term order yields

$$\begin{aligned}\Theta^{(0)}(\chi)\,I &= I && \text{(V.27)}\\ \Theta^{(0)}(\chi)\,\Phi^{(1)}(\chi) + \Theta^{(1)}(\chi)\,I &= 0 && \text{(V.28)}\\ \Theta^{(1)}(\chi)\,\Phi^{(1)}(\chi) + \Theta^{(2)}(\chi)\,I &= 0 && \text{(V.29)}\\ \Theta^{(2)}(\chi)\,\Phi^{(1)}(\chi) + \Theta^{(3)}(\chi)\,I &= 0 && \text{(V.30)}\\ &\vdots\end{aligned}$$

which, when rearranged, yields the expected result. ◁

So, applying the results of the lemma to our problem yields

$$\left(I+\frac{\partial}{\partial\chi}h^{(2)}(\chi)\right)^{-1}=I-\frac{\partial}{\partial\chi}h^{(2)}(\chi)+\left(-\frac{\partial}{\partial\chi}h^{(2)}(\chi)\right)^{2}+\left(-\frac{\partial}{\partial\chi}h^{(2)}(\chi)\right)^{3}+O^{(4+)} \quad (V.31)$$

and plugging this result into equation V.19 gives

$$\begin{aligned}\dot{\chi} &= F\chi+G\tilde{u} && (V.32)\\ &+ \left(\tilde{f}^{(2)}(\chi)+Fh^{(2)}(\chi)-\frac{\partial}{\partial\chi}h^{(2)}(\chi)F\chi\right) && (V.33)\\ &+ \left(\tilde{g}^{(1)}(\chi)-\frac{\partial}{\partial\chi}h^{(2)}(\chi)G\right)\tilde{u}\\ &+ \vec{f}^{(3)}(\chi)+\vec{g}^{(2)}(\chi)\tilde{u}+O^{(4+)}\end{aligned}$$

where we have defined

$$\vec{f}^{(3)}(\chi) = \tilde{f}^{(3)}(\chi)-\frac{\partial}{\partial\chi}h^{(2)}(\chi)\left(\tilde{f}^{(2)}(\chi)+Fh^{(2)}(\chi)-\frac{\partial}{\partial\chi}h^{(2)}(\chi)F\chi\right) \quad (V.34)$$

$$\vec{g}^{(2)}(\chi) = \tilde{g}^{(2)}(\chi)-\frac{\partial}{\partial\chi}h^{(2)}(\chi)\left(\tilde{g}^{(1)}(\chi)-\frac{\partial}{\partial\chi}h^{(2)}(\chi)G\right) \quad (V.35)$$

Temporarily neglecting the cubic terms for clarity, we can write our preliminary quadratic normal form as

$$\dot{\chi}=F\chi+G\tilde{u}+\check{f}^{(2)}(\chi)+\check{g}^{(1)}(\chi)\tilde{u}+O^{(3+)} \quad (V.36)$$

where we have defined our quadratic normal form terms as

$$\check{f}^{(2)}(\chi)=\tilde{f}^{(2)}(\chi)+Fh^{(2)}(\chi)-\frac{\partial}{\partial\chi}h^{(2)}(\chi)F\chi \quad (V.37)$$

and

$$\check{g}^{(1)}(\chi)=\tilde{g}^{(1)}(\chi)-\frac{\partial}{\partial\chi}h^{(2)}(\chi)G \quad (V.38)$$

Equation V.37 is known as the homological equation, while equation V.38 does not have as commonly accepted a name, although it has also been referred to as a homological equation [Ref. 2]. However, in this dissertation we will refer to equation V.38 as the constraint equation, for reasons which will become clear later. We will use equations V.37 and V.38 to eliminate as many terms of $\check{f}^{(2)}(\chi)$ and $\check{g}^{(1)}(\chi)$ as possible by properly picking $h^{(2)}(\chi)$.

2. Application of Regular Feedback

However, before we examine the effect of picking $h^{(2)}(\chi)$ on $\check{f}^{(2)}(\chi)$ and $\check{g}^{(1)}(\chi)$, we should look at the effect of applying feedback. Looking at equation V.36, we see that the simple expedient of defining our new control input after feedback has been applied to be the bottom component of the equation, can eliminate many terms of $\check{f}^{(2)}(\chi)$ and $\check{g}^{(1)}(\chi)$ without considering whether a coordinate transformation has been applied or not. That is, if

$$\check{f}^{(2)}(\chi) = \begin{bmatrix} \check{f}_1^{(2)}(\chi) \\ \vdots \\ \check{f}_{\nu-1}^{(2)}(\chi) \\ \check{f}_{\nu}^{(2)}(\chi) \end{bmatrix} \tag{V.39}$$

and

$$\check{g}^{(1)}(\chi) = \begin{bmatrix} \check{g}_1^{(1)}(\chi) \\ \vdots \\ \check{g}_{\nu-1}^{(1)}(\chi) \\ \check{g}_{\nu}^{(1)}(\chi) \end{bmatrix} \tag{V.40}$$

and if we define our new control input after feedback, v, as

$$v = \tilde{u} + \check{f}_{\nu}^{(2)}(\chi) + \check{g}_{\nu}^{(1)}(\chi)\,\tilde{u} \tag{V.41}$$

then the bottom row of equation V.36 becomes pure control, i.e.

$$\dot{\chi}_{\nu} = v + O^{(3+)} \tag{V.42}$$

The practical effect of applying equation V.41 is to remove the requirement that the coordinate transformation eliminate terms in the bottom row — the coordinate transformation can now be allowed to alter the bottom row arbitrarily, and the terms can still be eliminated by applying the feedback of equation V.41. We summarize this result in a theorem.

Theorem B.2 (Non-Linear Feedback) *For a dynamic control system in the form of equation V.36, with $\chi \in R^{\nu}$, the non-linear control law*

$$\tilde{u} = v - \check{g}_{\nu}^{(1)}(\chi)\, v - \check{f}_{\nu}^{(2)}(\chi) \tag{V.43}$$

suffices to eliminate the quadratic order terms of the bottom row (ν component).

Proof. We prove the theorem by direct calculation. The bottom row (ν component) of the system in equation V.36, is given by

$$\dot{\chi}_{\nu} = \tilde{u} + \check{f}_{\nu}^{(2)}(\chi) + \check{g}_{\nu}^{(1)}(\chi)\, \tilde{u} + O^{(3+)} \tag{V.44}$$

Plugging in our control law, we have

$$\begin{aligned} \dot{\chi}_{\nu} &= v - \check{g}_{\nu}^{(1)}(\chi)\, v - \check{f}_{\nu}^{(2)}(\chi) \\ &\quad + \check{f}_{\nu}^{(2)}(\chi) + \check{g}_{\nu}^{(1)}(\chi) \left(v - \check{g}_{\nu}^{(1)}(\chi)\, v - \check{f}_{\nu}^{(2)}(\chi)\right) \\ &= v - \check{g}_{\nu}^{(1)}(\chi) \left(\check{g}_{\nu}^{(1)}(\chi)\, v + \check{f}_{\nu}^{(2)}(\chi)\right) + O^{(3+)} \\ &= v + O^{(3+)} \end{aligned} \tag{V.45}$$

which proves the theorem. ◃

3. Solving the Constraint Equation

Now that we know that the bottom row of our system can be eliminated with feedback after our coordinate transformation is complete, we need to choose our transformation, $h^{(2)}(\chi)$. There are two problems we are trying to solve. First, pick $h^{(2)}(\chi)$ so as to eliminate as many terms of $\check{f}^{(2)}(\chi)$ as possible in equation V.37 (except in the bottom row, which will be handled by feedback). Second, pick $h^{(2)}(\chi)$ to eliminate as many terms of $\check{g}^{(1)}(\chi)$ as possible in equation V.38 (except again the bottom row, which will be handled by feedback). Now, in general, we cannot eliminate all terms in both $\check{f}^{(2)}(\chi)$ and $\check{g}^{(1)}(\chi)$ simultaneously, because there are not enough degrees of freedom in $h^{(2)}(\chi)$. Also, we find that regardless of how many degrees of freedom we have, there are sometimes terms in $\check{f}^{(2)}(\chi)$ which we cannot eliminate at all (called resonant terms). However, we find that we can always eliminate all

the terms in $\check{g}^{(1)}(\chi)$. So our strategy is to pick $h^{(2)}(\chi)$ to eliminate all the terms in $\check{g}^{(1)}(\chi)$, and then use the remaining terms to eliminate as many terms of $\check{f}^{(2)}(\chi)$ as possible. We state our first result in a theorem.

Theorem B.3 (Elimination of g(x)) *For any control system of the form of equation V.11 rewritten here as*

$$\begin{aligned}\dot{\tilde{\chi}} &= F\tilde{\chi} + G\tilde{u} + \tilde{f}^{(2)}(\tilde{\chi}) + \tilde{g}^{(1)}(\tilde{\chi})\,\tilde{u} \\ &+ \tilde{f}^{(3)}(\tilde{\chi}) + \tilde{g}^{(2)}(\tilde{\chi})\,\tilde{u} + O^{(4+)}\end{aligned} \tag{V.46}$$

with $\tilde{\chi} \in R^{\nu}$, and with G of the form

$$G = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{V.47}$$

a coordinate change of the form of equation V.12 completely eliminates the term $\tilde{g}^{(1)}(\tilde{\chi})$ if the elements of $h^{(2)}(\chi)$ are chosen such that

$$h_i(\chi) = \int \tilde{g}_i^{(1)}(\chi)\, d\chi_\nu + \varphi_i^{(2)}(\chi_1, \ldots, \chi_{\nu-1}) \tag{V.48}$$

for $i = 1$ to $\nu - 1$, where

$$h^{(2)}(\chi) = \begin{bmatrix} h_1(\chi) \\ \vdots \\ h_\nu(\chi) \end{bmatrix} \tag{V.49}$$

and where $\varphi_i^{(2)}(\chi_1, \ldots, \chi_{\nu-1})$ are arbitrary quadratic functions. The element $h_\nu(\chi)$ may be chosen as an arbitrary function if feedback is used to eliminate the term $\tilde{g}_\nu^{(1)}(\chi)$.

Proof. We prove the theorem by direct calculation. The transformed control term, $\check{g}^{(1)}(\chi)$, is given by equation V.38, which we write as

$$\begin{bmatrix} \check{g}_1^{(1)}(\chi) \\ \vdots \\ \check{g}_{\nu-1}^{(1)}(\chi) \\ \check{g}_\nu^{(1)}(\chi) \end{bmatrix} = \begin{bmatrix} \tilde{g}_1^{(1)}(\chi) \\ \vdots \\ \tilde{g}_{\nu-1}^{(1)}(\chi) \\ \tilde{g}_\nu^{(1)}(\chi) \end{bmatrix} - \begin{bmatrix} \frac{\partial h_1}{\partial \chi_1} & \cdots & \frac{\partial h_1}{\partial \chi_{\nu-1}} & \frac{\partial h_1}{\partial \chi_\nu} \\ \vdots & & \vdots & \vdots \\ \frac{\partial h_{\nu-1}}{\partial \chi_1} & \cdots & \frac{\partial h_{\nu-1}}{\partial \chi_{\nu-1}} & \frac{\partial h_{\nu-1}}{\partial \chi_\nu} \\ \frac{\partial h_\nu}{\partial \chi_1} & \cdots & \frac{\partial h_\nu}{\partial \chi_{\nu-1}} & \frac{\partial h_\nu}{\partial \chi_\nu} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{V.50}$$

$$
= \begin{bmatrix} \tilde{g}_1^{(1)}(\chi) \\ \vdots \\ \tilde{g}_{\nu-1}^{(1)}(\chi) \\ \tilde{g}_\nu^{(1)}(\chi) \end{bmatrix} - \begin{bmatrix} \frac{\partial h_1}{\partial \chi_\nu} \\ \vdots \\ \frac{\partial h_{\nu-1}}{\partial \chi_\nu} \\ \frac{\partial h_\nu}{\partial \chi_\nu} \end{bmatrix}
$$

Now, if we set $\check{g}_i^{(1)}(\chi) = 0$ for any component equation, we get equation V.48, which proves the theorem. Note that since components of $\check{g}^{(1)}(\chi)$ can be killed off individually by components of $h^{(2)}(\chi)$, we can choose to not eliminate the bottom component of $\check{g}^{(1)}(\chi)$, since that can be eliminated with feedback, which saves the bottom component of $h^{(2)}(\chi)$ for use in eliminating terms of $\check{f}^{(2)}(\chi)$. ◁

4. Solving the Homological Equation

Now we consider how to eliminate terms of $\check{f}^{(2)}(\chi)$ in equation V.37. We have two competing requirements. First, we need to show the general method of solution, which will lead to the Unstacking Theorem. Second, we need to exploit the structure of the problem to break it up into smaller problems, which lend themselves to individual solution. This will lead to the separation theorem. We begin with the general method of solution. Rewriting the terms in equation V.37 in vector/matrix form as

$$\check{f}^{(2)}(\chi) = \check{Q}\chi^{(2)} \tag{V.51}$$

$$\tilde{f}^{(2)}(\chi) = \tilde{Q}\chi^{(2)} \tag{V.52}$$

$$h^{(2)}(\chi) = H\chi^{(2)} \tag{V.53}$$

where $\check{Q}$, $\tilde{Q}$ and H are matrices of coefficients and $\chi^{(2)}$ is the quadratic appended state vector in the form given by the lemma "Notation for Multi-Variable Taylor Series Expansions" from Chapter III. We can write equation V.37 as

$$\check{Q}\chi^{(2)} = \tilde{Q}\chi^{(2)} + FH\chi^{(2)} - H\frac{\partial}{\partial\chi}\chi^{(2)}F\chi \tag{V.54}$$

Now, the last term in this equation is of special interest. Using the method discussed in detail in Appendix A, we can define a matrix of coefficients, D, by the relation

$$D\chi^{(2)} \equiv \frac{\partial}{\partial\chi}\chi^{(2)}F\chi \tag{V.55}$$

(Notice that D depends only on the component order in the quadratic state vector $\chi^{(2)}$, and on the structure of the matrix F, but is otherwise independent of the specifics of the problem at hand. For this reason we refer to matrices such as D as structural matrices.) We can write equation V.54 as

$$\check{Q}\chi^{(2)} = \left(\tilde{Q} + FH - HD\right)\chi^{(2)} \tag{V.56}$$

which is only true when

$$HD - FH = \tilde{Q} - \check{Q} \tag{V.57}$$

This brings us to the Unstacking Theorem.

Theorem B.4 (The Unstacking Theorem) *Any matrix equation of the form of equation V.57 can be converted into the vector matrix equation*

$$\left(\begin{bmatrix} D^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D^T \end{bmatrix} - \begin{bmatrix} F_{11}I & \cdots & \cdots & F_{1\nu}I \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ F_{\nu 1}I & \cdots & \cdots & F_{\nu\nu}I \end{bmatrix}\right)\begin{bmatrix} H_1^T \\ \vdots \\ \vdots \\ H_\nu^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_1^T \\ \vdots \\ \vdots \\ \tilde{Q}_\nu^T \end{bmatrix} - \begin{bmatrix} \check{Q}_1^T \\ \vdots \\ \vdots \\ \check{Q}_\nu^T \end{bmatrix} \tag{V.58}$$

which can be used to find a solution for H and $\check{Q}$. The component matrices are

$$D, I \in R^{\frac{\nu(\nu+1)}{2} \times \frac{\nu(\nu+1)}{2}} \tag{V.59}$$

$$F = \begin{bmatrix} F_{11} & \cdots & F_{1\nu} \\ \vdots & & \vdots \\ F_{\nu 1} & \cdots & F_{\nu\nu} \end{bmatrix} \in R^{\nu\times\nu} \tag{V.60}$$

$$H = \begin{bmatrix} H_1 \\ \vdots \\ H_\nu \end{bmatrix} \in R^{\nu\times\frac{\nu(\nu+1)}{2}} \tag{V.61}$$

$$\tilde{Q} = \begin{bmatrix} \tilde{Q}_1 \\ \vdots \\ \tilde{Q}_\nu \end{bmatrix} \in R^{\nu\times\frac{\nu(\nu+1)}{2}} \tag{V.62}$$

$$\check{Q} = \begin{bmatrix} \check{Q}_1 \\ \vdots \\ \check{Q}_\nu \end{bmatrix} \in R^{\nu \times \frac{\nu(\nu+1)}{2}} \tag{V.63}$$

where the notation H_i etc. has been used to indicate the ith row of the particular matrix, the notation F_{ij} has been used to indicate the ijth element of the particular matrix, and where I indicates the identity matrix.

Proof. Given an equation of the form

$$HD - FH = \tilde{Q} - \check{Q} \tag{V.64}$$

with matrices defined as in equations V.59 through V.63, look at the equation row-by-row. We have

$$\left(\begin{bmatrix} H_1 D \\ \vdots \\ H_\nu D \end{bmatrix} - \begin{bmatrix} F_{11}H_1 + \ldots + F_{1\nu}H_\nu \\ \vdots \\ F_{\nu 1}H_1 + \ldots + F_{\nu\nu}H_1 \end{bmatrix} \right) = \begin{bmatrix} \tilde{Q}_1 \\ \vdots \\ \tilde{Q}_\nu \end{bmatrix} - \begin{bmatrix} \check{Q}_1 \\ \vdots \\ \check{Q}_\nu \end{bmatrix} \tag{V.65}$$

Unstacking the rows of the matrices to the left to form one long row vector yields

$$\begin{aligned} & \begin{bmatrix} H_1 D & \cdots & H_\nu D \end{bmatrix} \\ - & \begin{bmatrix} (F_{11}H_1 + \ldots + F_{1\nu}H_\nu) & \cdots & (F_{\nu 1}H_1 + \ldots + F_{\nu\nu}H_1) \end{bmatrix} \\ = & \begin{bmatrix} \tilde{Q}_1 & \cdots & \tilde{Q}_\nu \end{bmatrix} - \begin{bmatrix} \check{Q}_1 & \cdots & \check{Q}_\nu \end{bmatrix} \end{aligned} \tag{V.66}$$

which when transposed and restacked yields the desired result

$$\left(\begin{bmatrix} D^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D^T \end{bmatrix} - \begin{bmatrix} F_{11}I & \cdots & \cdots & F_{1\nu}I \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ F_{\nu 1}I & \cdots & \cdots & F_{\nu\nu}I \end{bmatrix} \right) \begin{bmatrix} H_1^T \\ \vdots \\ \vdots \\ H_\nu^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_1^T \\ \vdots \\ \vdots \\ \tilde{Q}_\nu^T \end{bmatrix} - \begin{bmatrix} \check{Q}_1^T \\ \vdots \\ \vdots \\ \check{Q}_\nu^T \end{bmatrix} \tag{V.67}$$

◁

Corollary B.5 (The Unstacking Corollary) *Any matrix equation of the form*

$$HD = \tilde{Q} - \check{Q} \tag{V.68}$$

can be converted into the vector/matrix equation

$$\begin{bmatrix} D^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D^T \end{bmatrix} \begin{bmatrix} H_1^T \\ \vdots \\ \vdots \\ H_\nu^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_1^T \\ \vdots \\ \vdots \\ \tilde{Q}_\nu^T \end{bmatrix} - \begin{bmatrix} \check{Q}_1^T \\ \vdots \\ \vdots \\ \check{Q}_\nu^T \end{bmatrix} \tag{V.69}$$

with the notational conventions the same as the theorem.

Proof. This is just a special case of the theorem result with $F = 0$. ◁

5. The Separation Principle

Now look at our system in linear normal form again, given by equation V.2. The appended state vector is

$$\begin{bmatrix} \mu \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} \tag{V.70}$$

where μ is the vector of parameters, $\tilde{z}$ is the vector of linearly uncontrollable states with zero real-part eigenvalues, $\tilde{w}$ is the vector of linearly uncontrollable states with non-zero real-part eigenvalues, and $\tilde{y}$ is the vector of linearly controllable states. The states μ, $\tilde{z}$, and $\tilde{w}$ are all linearly uncontrollable, so if we define the state vector $\tilde{\sigma}$ in block form as

$$\tilde{\sigma} = \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \end{bmatrix} \tag{V.71}$$

then we can represent the entire appended state vector in block form as

$$\begin{bmatrix} \tilde{\sigma} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{w} \\ \tilde{y} \end{bmatrix} \tag{V.72}$$

In Chapter III, in the lemma entitled "Notation for Multi-Variable Taylor Series Expansions", the notation $\xi^{(2)}$ was introduced to stand for the vector containing all possible quadratic combinations of the elements of the vector ξ. For the case where two vectors, $\tilde{\sigma}$ and $\tilde{y}$, are involved, we need to introduce additional notation to account for the cross terms involved. We state this in the form of a lemma.

Lemma B.6 (Quadratic State Vector) *A state vector $\xi \in R^g$, made up of two component state vectors $\sigma \in R^j$ and $\rho \in R^k$, can be represented in block form as*

$$\xi = \begin{bmatrix} \sigma \\ \rho \end{bmatrix} \tag{V.73}$$

The vector $\xi^{(2)} \in R^{\frac{g(g+1)}{2}}$, containing all possible quadratic combinations of the elements of ξ, is called the quadratic state vector, and can be represented in block form as

$$\xi^{(2)} = \begin{bmatrix} \sigma^{(2)} \\ \sigma\rho^{(2)} \\ \rho^{(2)} \end{bmatrix} \tag{V.74}$$

The quadratic state vectors $\sigma^{(2)} \in R^{\frac{j(j+1)}{2}}$ and $\rho^{(2)} \in R^{\frac{k(k+1)}{2}}$ contain all the possible quadratic combinations of the elements of the linear state vectors σ and ρ respectively, as explained in the lemma "Notation for Multi-Variable Taylor Series Expansions" in Chapter III. However, new notation has been introduced for the quadratic mixed terms. The vector $\sigma\rho^{(2)} \in R^{jk}$ is called the mixed quadratic state vector, and contains all the possible quadratic combinations such that one element is from σ and one element is from ρ. The vector $\sigma\rho^{(2)}$ can be represented in block form as

$$\sigma\rho^{(2)} = \begin{bmatrix} \sigma\rho_1 \\ \vdots \\ \sigma\rho_k \end{bmatrix} \tag{V.75}$$

where each block is the vector σ multiplied by the appropriate element of ρ. The rule for ordering the elements $\xi_h\xi_i$ of each of the quadratic state vectors $\sigma^{(2)}$, $\sigma\rho^{(2)}$ and $\rho^{(2)}$ is $\xi_h\xi_i > \xi_s\xi_t$ if $i > t$, or $h > s$ if $i = t$. Then the smaller elements are stacked on top of the larger elements.

Proof. Although the ordering of the individual elements is just a matter of convention, we would like to show that, taken together, the quadratic state vectors $\sigma^{(2)}$, $\sigma\rho^{(2)}$ and $\rho^{(2)}$ include all possible quadratic combinations of the elements of

the linear state vector ξ, and include no extra terms. The vector ξ contains all the elements of σ and all the elements of ρ. The vector $\xi^{(2)}$ contains all the possible combinations of the elements of ξ taken two at a time, and there are only three ways to combine elements two at a time: both elements can be from σ, in which case the combination is an element of $\sigma^{(2)}$; one element can be from σ and one element can be from ρ, in which case the combination is an element of $\sigma\rho^{(2)}$; or both elements can be from ρ, in which case the combination is an element of $\rho^{(2)}$. So, $\xi^{(2)}$ is given by equation V.74. $\triangleleft$

Now we desire to simplify our problem by breaking it up into smaller pieces which can be solved individually, and then recombined to yield the full solution. We begin by exploiting the structure of F, which we write as

$$F = \begin{bmatrix} F_\sigma & 0 \\ 0 & A \end{bmatrix} \tag{V.76}$$

Pulling the linearly controllable states out of our state vector χ, we get

$$\chi = \begin{bmatrix} \sigma \\ y \end{bmatrix} \tag{V.77}$$

In both cases we have used the definitions

$$F_\sigma = \begin{bmatrix} 0 & 0 & 0 \\ F_\mu & F_z & 0 \\ 0 & 0 & F_w \end{bmatrix} \tag{V.78}$$

and

$$\sigma = \begin{bmatrix} \mu \\ z \\ w \end{bmatrix} \tag{V.79}$$

Finally, we separate out the linearly controllable/uncontrollable components of $\check{f}^{(2)}(\chi)$, $\tilde{f}^{(2)}(\chi)$ and $h^{(2)}(\chi)$ in block form as

$$\check{f}^{(2)}(\chi) = \begin{bmatrix} \check{f}^{(2)}_\sigma(\sigma, y) \\ \check{f}^{(2)}_y(\sigma, y) \end{bmatrix} \tag{V.80}$$

$$\tilde{f}^{(2)}(\chi) = \begin{bmatrix} \tilde{f}_{\sigma}^{(2)}(\sigma, y) \\ \tilde{f}_{y}^{(2)}(\sigma, y) \end{bmatrix} \tag{V.81}$$

$$h^{(2)}(\chi) = \begin{bmatrix} h_{\sigma}^{(2)}(\sigma, y) \\ h_{y}^{(2)}(\sigma, y) \end{bmatrix} \tag{V.82}$$

Pluggging our block forms into equation V.37, gives

$$\begin{bmatrix} \check{f}_{\sigma}^{(2)}(\sigma, y) \\ \check{f}_{y}^{(2)}(\sigma, y) \end{bmatrix} = \begin{bmatrix} \tilde{f}_{\sigma}^{(2)}(\sigma, y) \\ \tilde{f}_{y}^{(2)}(\sigma, y) \end{bmatrix} + \begin{bmatrix} F_{\sigma} & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} h_{\sigma}^{(2)}(\sigma, y) \\ h_{y}^{(2)}(\sigma, y) \end{bmatrix} \tag{V.83}$$

$$- \begin{bmatrix} \frac{\partial h_{\sigma}}{\partial \sigma} & \frac{\partial h_{\sigma}}{\partial y} \\ \frac{\partial h_{y}}{\partial \sigma} & \frac{\partial h_{y}}{\partial y} \end{bmatrix} \begin{bmatrix} F_{\sigma} & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \sigma \\ y \end{bmatrix} \tag{V.84}$$

Using the previous technique of rewriting the terms in equation V.37 in vector/matrix form as

$$\check{f}^{(2)}(\chi) = \check{Q}\chi^{(2)} \tag{V.85}$$

$$\tilde{f}^{(2)}(\chi) = \tilde{Q}\chi^{(2)} \tag{V.86}$$

$$h^{(2)}(\chi) = H\chi^{(2)} \tag{V.87}$$

we can write equation V.83 as

$$\begin{bmatrix} \check{Q}_{\sigma u} & \check{Q}_{\sigma m} & \check{Q}_{\sigma c} \\ \check{Q}_{yu} & \check{Q}_{ym} & \check{Q}_{yc} \end{bmatrix} \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{\sigma u} & \tilde{Q}_{\sigma m} & \tilde{Q}_{\sigma c} \\ \tilde{Q}_{yu} & \tilde{Q}_{ym} & \tilde{Q}_{yc} \end{bmatrix} \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix} \tag{V.88}$$

$$+ \begin{bmatrix} F_{\sigma} & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} H_{\sigma u} & H_{\sigma m} & H_{\sigma c} \\ H_{yu} & H_{ym} & H_{yc} \end{bmatrix} \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix}$$

$$- \begin{bmatrix} H_{\sigma u} & H_{\sigma m} & H_{\sigma c} \\ H_{yu} & H_{ym} & H_{yc} \end{bmatrix} \begin{bmatrix} D_{\sigma} & 0 & 0 \\ 0 & D_m & 0 \\ 0 & 0 & D_c \end{bmatrix} \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix}$$

where we have used the results of the "Quadratic State Vector" lemma, and where the subscripts u, m and c stand for "uncontrollable", "mixed" and "controllable"

respectively, with the reference to controllability referring to linear controllability. We have also used the definitions for D_σ, D_m and D_c given in Appendix A as

$$D_\sigma \sigma^{(2)} = \frac{\partial \sigma^{(2)}}{\partial \sigma} F_\sigma \sigma \tag{V.89}$$

$$D_m \sigma y^{(2)} = \frac{\partial \sigma y^{(2)}}{\partial \sigma} F_\sigma \sigma + \frac{\partial \sigma y^{(2)}}{\partial y} A y \tag{V.90}$$

$$D_c y^{(2)} = \frac{\partial y^{(2)}}{\partial y} A y \tag{V.91}$$

Equation V.88 is only true if the coefficient matrices satisfy the relation

$$\begin{aligned} &\begin{bmatrix} H_{\sigma u} & H_{\sigma m} & H_{\sigma c} \\ H_{yu} & H_{ym} & H_{yc} \end{bmatrix} \begin{bmatrix} D_\sigma & 0 & 0 \\ 0 & D_m & 0 \\ 0 & 0 & D_c \end{bmatrix} \\ &- \begin{bmatrix} F_\sigma & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} H_{\sigma u} & H_{\sigma m} & H_{\sigma c} \\ H_{yu} & H_{ym} & H_{yc} \end{bmatrix} \\ = &\begin{bmatrix} \tilde{Q}_{\sigma u} & \tilde{Q}_{\sigma m} & \tilde{Q}_{\sigma c} \\ \tilde{Q}_{yu} & \tilde{Q}_{ym} & \tilde{Q}_{yc} \end{bmatrix} - \begin{bmatrix} \check{Q}_{\sigma u} & \check{Q}_{\sigma m} & \check{Q}_{\sigma c} \\ \check{Q}_{yu} & \check{Q}_{ym} & \check{Q}_{yc} \end{bmatrix} \end{aligned} \tag{V.92}$$

which is in the form needed for the Unstacking Theorem. But, because of the block diagonal nature of the equation, it can actually be solved as six uncoupled equations. We state our result as the "Separation Principle" theorem.

Theorem B.7 (Separation Principle) *When the linear state matrix F has the block diagonal form*

$$F = \begin{bmatrix} F_\sigma & 0 \\ 0 & A \end{bmatrix} \tag{V.93}$$

and the state vector χ is given in block form by

$$\chi = \begin{bmatrix} \sigma \\ y \end{bmatrix} \tag{V.94}$$

the solutions $\check{f}^{(2)}(\chi)$ and $h^{(2)}(\chi)$ to the homological equation

$$\check{f}^{(2)}(\chi) = \tilde{f}^{(2)}(\chi) + F h^{(2)}(\chi) - \frac{\partial}{\partial \chi} h^{(2)}(\chi) F \chi \tag{V.95}$$

are given by

$$\check{f}^{(2)}(\chi) = \begin{bmatrix} \check{Q}_{\sigma u} & \check{Q}_{\sigma m} & \check{Q}_{\sigma c} \\ \check{Q}_{yu} & \check{Q}_{ym} & \check{Q}_{yc} \end{bmatrix} \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix} \tag{V.96}$$

$$h^{(2)}(\chi) = \begin{bmatrix} H_{\sigma u} & H_{\sigma m} & H_{\sigma c} \\ H_{yu} & H_{ym} & H_{yc} \end{bmatrix} \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix} \tag{V.97}$$

With $\tilde{f}^{(2)}(\chi)$ *given by*

$$\tilde{f}^{(2)}(\chi) = \begin{bmatrix} \tilde{Q}_{\sigma u} & \tilde{Q}_{\sigma m} & \tilde{Q}_{\sigma c} \\ \tilde{Q}_{yu} & \tilde{Q}_{ym} & \tilde{Q}_{yc} \end{bmatrix} \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix} \tag{V.98}$$

the block elements of the coefficient matrices

$$\check{Q} = \begin{bmatrix} \check{Q}_{\sigma u} & \check{Q}_{\sigma m} & \check{Q}_{\sigma c} \\ \check{Q}_{yu} & \check{Q}_{ym} & \check{Q}_{yc} \end{bmatrix} \tag{V.99}$$

$$H = \begin{bmatrix} H_{\sigma u} & H_{\sigma m} & H_{\sigma c} \\ H_{yu} & H_{ym} & H_{yc} \end{bmatrix} \tag{V.100}$$

are found as solutions of the separable matrix equations

$$H_{yc}D_c - AH_{yc} = \tilde{Q}_{yc} - \check{Q}_{yc} \tag{V.101}$$

$$H_{ym}D_m - AH_{ym} = \tilde{Q}_{ym} - \check{Q}_{ym} \tag{V.102}$$

$$H_{yu}D_\sigma - AH_{yu} = \tilde{Q}_{yu} - \check{Q}_{yu} \tag{V.103}$$

$$H_{\sigma c}D_c - F_\sigma H_{\sigma c} = \tilde{Q}_{\sigma c} - \check{Q}_{\sigma c} \tag{V.104}$$

$$H_{\sigma m}D_m - F_\sigma H_{\sigma m} = \tilde{Q}_{\sigma m} - \check{Q}_{\sigma m} \tag{V.105}$$

$$H_{\sigma u}D_\sigma - F_\sigma H_{\sigma u} = \tilde{Q}_{\sigma u} - \check{Q}_{\sigma u} \tag{V.106}$$

which individually have the form needed to apply the Unstacking Theorem for solution.

Proof. Plugging equations V.93, V.96, V.97 and V.98 into equation V.95, and using the definitions of D_σ, D_m and D_c given in equations V.89, V.90 and V.91 gives an equation of the form V.88, which can only be true if the coefficient equation V.92 is true. When equation V.92 is multiplied out we get

$$\begin{bmatrix} H_{\sigma u}D_\sigma - F_\sigma H_{\sigma u} & H_{\sigma m}D_m - F_\sigma H_{\sigma m} & H_{\sigma c}D_c - F_\sigma H_{\sigma c} \\ H_{yu}D_\sigma - AH_{yu} & H_{ym}D_m - AH_{ym} & H_{yc}D_c - AH_{yc} \end{bmatrix} \tag{V.107}$$
$$= \begin{bmatrix} \tilde{Q}_{\sigma u} - \check{Q}_{\sigma u} & \tilde{Q}_{\sigma m} - \check{Q}_{\sigma m} & \tilde{Q}_{\sigma c} - \check{Q}_{\sigma c} \\ \tilde{Q}_{yu} - \check{Q}_{yu} & \tilde{Q}_{ym} - \check{Q}_{ym} & \tilde{Q}_{yc} - \check{Q}_{yc} \end{bmatrix}$$

which is only true if equations V.101 through V.106 are true individually. ◁

Now, we have determined how to find a solution for the coefficients of our quadratic coordinate transformation, $h^{(2)}(\chi)$, and our quadratic normal form function, $\check{f}^{(2)}(\chi)$, using the Separation Principle theorem and the Unstacking Theorem. However, we have not yet taken into account the fact that part of $h^{(2)}(\chi)$ was used to eliminate the $\tilde{g}^{(1)}(\chi)$ term, nor the fact that our quadratic order state feedback can eliminate terms in $\check{f}^{(2)}(\chi)$ without using any terms in $h^{(2)}(\chi)$. These additional conditions act like constraints or relief from constraints on our search for solutions to equations V.101 through V.106. We summarize these conditions in two lemmas.

Lemma B.8 (Constraints on H from g(x)) *The search for solutions*

$$h^{(2)}(\chi) = \begin{bmatrix} H_{\sigma u} & H_{\sigma m} & H_{\sigma c} \\ H_{yu} & H_{ym} & H_{yc} \end{bmatrix} \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix} \tag{V.108}$$

to equations V.101 through V.106 is constrained by the fact that elements of $h^{(2)}(\chi)$ were used to eliminate the $\tilde{g}^{(1)}(\chi)$ term, as detailed in the "Elimination of g(x)" theorem. These constraints may be stated as the block matrix equations

$$\tilde{G}_{\sigma u} - H_{\sigma m} D_{B_\sigma} = 0 \tag{V.109}$$

$$\tilde{G}_{\sigma c} - H_{\sigma c} D_{B_y} = 0 \tag{V.110}$$

$$\tilde{G}_{yu} - H_{ym} D_{B_\sigma} = \check{G}_{yu} \tag{V.111}$$

$$\tilde{G}_{yc} - H_{yc} D_{B_y} = \check{G}_{yc} \tag{V.112}$$

where

$$\tilde{g}^{(1)}(\chi) = \begin{bmatrix} \tilde{G}_{\sigma u} & \tilde{G}_{\sigma c} \\ \tilde{G}_{yu} & \tilde{G}_{yc} \end{bmatrix} \begin{bmatrix} \sigma \\ y \end{bmatrix} \tag{V.113}$$

and where the structural matrices D_{B_σ} and D_{B_y} are defined in Appendix A by the relations

$$D_{B_\sigma} \sigma = \frac{\partial \sigma y^{(2)}}{\partial y} B \tag{V.114}$$

$$D_{B_y} y = \frac{\partial y^{(2)}}{\partial y} B \tag{V.115}$$

The matrices $\check{G}_{yu}$ and $\check{G}_{yc}$ have the form

$$\check{G}_{yu} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \check{G}_{yu_p} \end{bmatrix} \tag{V.116}$$

$$\check{G}_{yc} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \check{G}_{yc_p} \end{bmatrix} \tag{V.117}$$

where the subscript p indicates the pth row.

Proof. Equation V.38, which we rewrite here for convenience as

$$\check{g}^{(1)}(\chi) = \tilde{g}^{(1)}(\chi) - \frac{\partial}{\partial \chi} h^{(2)}(\chi) G \tag{V.118}$$

can be put into block vector/matrix form

$$\check{g}^{(1)}(\chi) = \begin{bmatrix} \tilde{G}_{\sigma u} & \tilde{G}_{\sigma c} \\ \tilde{G}_{yu} & \tilde{G}_{yc} \end{bmatrix} \begin{bmatrix} \sigma \\ y \end{bmatrix} - \begin{bmatrix} H_{\sigma u} & H_{\sigma m} & H_{\sigma c} \\ H_{yu} & H_{ym} & H_{yc} \end{bmatrix} \begin{bmatrix} \frac{\partial \sigma^{(2)}}{\partial \sigma} & 0 \\ \frac{\partial \sigma y^{(2)}}{\partial \sigma} & \frac{\partial \sigma y^{(2)}}{\partial y} \\ 0 & \frac{\partial y^{(2)}}{\partial y} \end{bmatrix} \begin{bmatrix} 0 \\ B \end{bmatrix} \tag{V.119}$$

where we have rewritten equation V.10 in block form as

$$G = \begin{bmatrix} 0 \\ B \end{bmatrix} \tag{V.120}$$

Multiplying out equation V.119 gives

$$\begin{aligned} \check{g}^{(1)}(\chi) &= \begin{bmatrix} \tilde{G}_{\sigma u} & \tilde{G}_{\sigma c} \\ \tilde{G}_{yu} & \tilde{G}_{yc} \end{bmatrix} \begin{bmatrix} \sigma \\ y \end{bmatrix} - \begin{bmatrix} H_{\sigma m} \frac{\partial \sigma y^{(2)}}{\partial y} B + H_{\sigma c} \frac{\partial y^{(2)}}{\partial y} B \\ H_{ym} \frac{\partial \sigma y^{(2)}}{\partial y} B + H_{yc} \frac{\partial y^{(2)}}{\partial y} B \end{bmatrix} \\ &= \begin{bmatrix} \tilde{G}_{\sigma u} - H_{\sigma m} D_{B_\sigma} & \tilde{G}_{\sigma c} - H_{\sigma c} D_{B_y} \\ \tilde{G}_{yu} - H_{ym} D_{B_\sigma} & \tilde{G}_{yc} - H_{yc} D_{B_y} \end{bmatrix} \begin{bmatrix} \sigma \\ y \end{bmatrix} \end{aligned} \tag{V.121}$$

where we have used the definitions for D_{B_σ} and D_{B_y} from equations V.114 and V.115. From the "Elimination of g(x)" theorem we have that any component of $\check{g}^{(1)}(\chi)$ can be eliminated by a choice of the corresponding component of $h^{(2)}(\chi)$,

$$h_i(\chi) = \int \tilde{g}_i^{(1)}(\chi)\, d\chi_\nu + \varphi_i(\chi_1, \ldots, \chi_{\nu-1}) \tag{V.122}$$

So, if we choose to eliminate every component of $\breve{g}^{(1)}(\chi)$ except for the last row, we get equations V.109 through V.112, which proves the lemma. $\triangleleft$

In the next lemma, we examine the effect of feedback on relieving constraints on $h^{(2)}(\chi)$.

Theorem B.9 (Feedback Relieves Constraints) *When the non-linear control law given by equation V.43 in the Non-Linear Feedback theorem is used to eliminate all quadratic order terms in the last row of the dynamic control system, the effect is to relieve some constraints on finding a solution for $h^{(2)}(\chi)$ in equations V.101, V.102 and V.103. Applying the Unstacking Theorem, then applying the control law of equation V.43 causes equations V.101, V.102 and V.103 to become*

$$\begin{bmatrix} D_c^T & -I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & D_c^T & -I \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} H_{yc_1}^T \\ \vdots \\ \vdots \\ H_{yc_{p-1}}^T \\ H_{yc_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{yc_1}^T - \check{Q}_{yc_1}^T \\ \vdots \\ \vdots \\ \tilde{Q}_{yc_{p-1}}^T - \check{Q}_{yc_{p-1}}^T \\ 0 \end{bmatrix} \tag{V.123}$$

$$\begin{bmatrix} D_m^T & -I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & D_m^T & -I \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} H_{ym_1}^T \\ \vdots \\ \vdots \\ H_{ym_{p-1}}^T \\ H_{ym_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{ym_1}^T - \check{Q}_{ym_1}^T \\ \vdots \\ \vdots \\ \tilde{Q}_{ym_{p-1}}^T - \check{Q}_{ym_{p-1}}^T \\ 0 \end{bmatrix} \tag{V.124}$$

and

$$\begin{bmatrix} D_\sigma^T & -I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & D_\sigma^T & -I \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} H_{yu_1}^T \\ \vdots \\ \vdots \\ H_{yu_{p-1}}^T \\ H_{yu_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{yu_1}^T - \check{Q}_{yu_1}^T \\ \vdots \\ \vdots \\ \tilde{Q}_{yu_{p-1}}^T - \check{Q}_{yu_{p-1}}^T \\ 0 \end{bmatrix} \tag{V.125}$$

Proof. When the Unstacking Theorem is applied to equations V.101, V.102 and V.103, they become

$$\left(\begin{bmatrix} D_c^T & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & D_c^T & 0 \\ 0 & \cdots & \cdots & 0 & D_c^T \end{bmatrix} - \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & I \\ 0 & & \cdots & & 0 \end{bmatrix} \right) \begin{bmatrix} H_{yc_1}^T \\ \vdots \\ \vdots \\ H_{yc_{p-1}}^T \\ H_{yc_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{yc_1}^T - \check{Q}_{yc_1}^T \\ \vdots \\ \vdots \\ \tilde{Q}_{yc_{p-1}}^T - \check{Q}_{yc_{p-1}}^T \\ \tilde{Q}_{yc_p}^T - \check{Q}_{yc_p}^T \end{bmatrix} \tag{V.126}$$

$$\left(\begin{bmatrix} D_m^T & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & D_m^T & 0 \\ 0 & \cdots & \cdots & 0 & D_m^T \end{bmatrix} - \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & I \\ 0 & & \cdots & & 0 \end{bmatrix}\right) \begin{bmatrix} H_{ym_1}^T \\ \vdots \\ \vdots \\ H_{ym_{p-1}}^T \\ H_{ym_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{ym_1}^T - \check{Q}_{ym_1}^T \\ \vdots \\ \vdots \\ \tilde{Q}_{ym_{p-1}}^T - \check{Q}_{ym_{p-1}}^T \\ \tilde{Q}_{ym_p}^T - \check{Q}_{ym_p}^T \end{bmatrix} \tag{V.127}$$

and

$$\left(\begin{bmatrix} D_\sigma^T & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & D_\sigma^T & 0 \\ 0 & \cdots & \cdots & 0 & D_\sigma^T \end{bmatrix} - \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & I \\ 0 & & \cdots & & 0 \end{bmatrix}\right) \begin{bmatrix} H_{yu_1}^T \\ \vdots \\ \vdots \\ H_{yu_{p-1}}^T \\ H_{yu_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{yu_1}^T - \check{Q}_{yu_1}^T \\ \vdots \\ \vdots \\ \tilde{Q}_{yu_{p-1}}^T - \check{Q}_{yu_{p-1}}^T \\ \tilde{Q}_{yu_p}^T - \check{Q}_{yu_p}^T \end{bmatrix} \tag{V.128}$$

When the non-linear control law given by equation V.43 is applied to these equations it has the effect of removing the bottom block row, which is equivalent to replacing all entries on the bottom block row with block zeroes. Consolidating the two square matrices then gives the desired result. ◃

Corollary B.10 *Applying the Unstacking Corollary, then applying the control law of equation V.43 causes the constraint equations V.111 and V.112 to become*

$$\begin{bmatrix} D_{B_\sigma}^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & D_{B_\sigma}^T & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} H_{ym_1}^T \\ \vdots \\ H_{ym_{p-1}}^T \\ H_{ym_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{yu_1}^T \\ \vdots \\ \tilde{G}_{yu_{p-1}}^T \\ 0 \end{bmatrix} \tag{V.129}$$

and

$$\begin{bmatrix} D_{B_y}^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & D_{B_y}^T & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} H_{yc_1}^T \\ \vdots \\ H_{yc_{p-1}}^T \\ H_{yc_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{yc_1}^T \\ \vdots \\ \tilde{G}_{yc_{p-1}}^T \\ 0 \end{bmatrix} \tag{V.130}$$

Proof. When the Unstacking Corollary is applied to equations V.111 and V.112, they become

$$\begin{bmatrix} D_{B_\sigma}^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & D_{B_\sigma}^T & 0 \\ 0 & \cdots & 0 & D_{B_\sigma}^T \end{bmatrix} \begin{bmatrix} H_{ym_1}^T \\ \vdots \\ H_{ym_{p-1}}^T \\ H_{ym_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{yu_1}^T \\ \vdots \\ \tilde{G}_{yu_{p-1}}^T \\ \tilde{G}_{yu_p}^T \end{bmatrix} - \begin{bmatrix} \check{G}_{yu_1}^T \\ \vdots \\ \check{G}_{yu_{p-1}}^T \\ \check{G}_{yu_p}^T \end{bmatrix} \tag{V.131}$$

and

$$\begin{bmatrix} D_{B_y}^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & D_{B_y}^T & 0 \\ 0 & \cdots & 0 & D_{B_y}^T \end{bmatrix} \begin{bmatrix} H_{yc_1}^T \\ \vdots \\ H_{yc_{p-1}}^T \\ H_{yc_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{yc_1}^T \\ \vdots \\ \tilde{G}_{yc_{p-1}}^T \\ \tilde{G}_{yc_p}^T \end{bmatrix} - \begin{bmatrix} \check{G}_{yc_1}^T \\ \vdots \\ \check{G}_{yc_{p-1}}^T \\ \check{G}_{yc_p}^T \end{bmatrix} \tag{V.132}$$

When the non-linear control law given by equation V.43 is applied to these equations it has the effect of removing the bottom block row, which is equivalent to replacing all entries on the bottom block row with block zeroes. Then, when we apply the "Elimination of g(x)" theorem, all the remaining terms of $\check{G}_{yu}$ and $\check{G}_{yc}$ can be set to zero. This gives the desired result. ◃

In the next section we will begin the final task of this chapter, which is to eliminate as many terms of $\check{f}^{(2)}(\chi)$ as possible through our choice of $h^{(2)}(\chi)$. We will find, that in certain cases, elimination of a particular term in $\check{f}^{(2)}(\chi)$ will not be possible at all. Such terms are called resonant terms. In other cases, elimination of certain terms will imply that elimination of certain other terms is not possible, and vice-versa. That is, we may be able to choose which terms to eliminate, but still not be able to eliminate every term. And finally, in certain cases we will discover that we have more degrees of freedom than required, and will be able to find several co-ordinate transformations which eliminate as many terms as possible. (And of course, combinations of these cases are also possible.)

C. NORMAL FORMS

In the previous sections we have proven several useful theorems, lemmas and corollaries. However, under all the detail hides a relatively simple problem we are trying to solve, and it is worth our time to briefly restate it here. We start with a dynamic control system in linear normal form, given by equation V.11, which we repeat for convenience

$$\begin{aligned} \tilde{\chi} &= F\tilde{\chi} + G\tilde{u} \\ &+ \tilde{f}^{(2)}(\tilde{\chi}) + \tilde{g}^{(1)}(\tilde{\chi})\,\tilde{u} \\ &+ \tilde{f}^{(3)}(\tilde{\chi}) + \tilde{g}^{(2)}(\tilde{\chi})\,\tilde{u} + O^{(4+)} \end{aligned} \tag{V.133}$$

A quadratic coordinate transformation which simplifies our quadratic terms without altering the form of our linear terms is given by equation V.12, which we repeat for convenience

$$\tilde{\chi} = \chi + h^{(2)}(\chi) \tag{V.134}$$

where $h^{(2)}(\chi)$ is an as-yet unknown quadratic function of χ, and χ is our transformed state vector. When we transform equation V.133 using equation V.134, we get

$$\begin{aligned} \dot{\chi} &= F\chi + G\tilde{u} \\ &+ \check{f}^{(2)}(\chi) + \check{g}^{(1)}(\chi)\,\tilde{u} \\ &+ \vec{f}^{(3)}(\chi) + \vec{g}^{(2)}(\chi)\,\tilde{u} + O^{(4+)} \end{aligned} \tag{V.135}$$

where we have defined our quadratic normal form terms as

$$\check{f}^{(2)}(\chi) = \tilde{f}^{(2)}(\chi) + Fh^{(2)}(\chi) - \frac{\partial}{\partial\chi}h^{(2)}(\chi)\,F\chi \tag{V.136}$$

and

$$\check{g}^{(1)}(\chi) = \tilde{g}^{(1)}(\chi) - \frac{\partial}{\partial\chi}h^{(2)}(\chi)\,G \tag{V.137}$$

and where we have defined the cubic terms as

$$\vec{f}^{(3)}(\chi) = \bar{f}^{(3)}(\chi) - \frac{\partial}{\partial\chi}h^{(2)}(\chi)\left(\tilde{f}^{(2)}(\chi) + Fh^{(2)}(\chi) - \frac{\partial}{\partial\chi}h^{(2)}(\chi)\,F\chi\right) \tag{V.138}$$

$$\vec{g}^{(2)}(\chi) = \bar{g}^{(2)}(\chi) - \frac{\partial}{\partial\chi}h^{(2)}(\chi)\left(\tilde{g}^{(1)}(\chi) - \frac{\partial}{\partial\chi}h^{(2)}(\chi)\,G\right) \tag{V.139}$$

with

$$
\begin{aligned}
\bar{f}^{(3)}(\chi) + O^{(4+)} &= \tilde{f}^{(3)}(\chi) + \left(\tilde{f}^{(2)}\left(\chi + h^{(2)}(\chi)\right) - \tilde{f}^{(2)}(\chi)\right) && \text{(V.140)} \\
\bar{g}^{(2)}(\chi) &= \tilde{g}^{(2)}(\chi) + \tilde{g}^{(1)}\left(h^{(2)}(\chi)\right) && \text{(V.141)}
\end{aligned}
$$

When we apply non-linear quadratic order feedback to equation V.135, using a control law of the form

$$
\tilde{u} = v - \check{g}_{\nu}^{(1)}(\chi)\, v - \check{f}_{\nu}^{(2)}(\chi) \qquad \text{(V.142)}
$$

where the subscript ν indicates the bottom row (ν-th component), we get

$$
\begin{aligned}
\dot{\chi} &= F\chi + Gv && \text{(V.143)} \\
&+ \left(\check{f}^{(2)}(\chi) - G\check{f}_{\nu}^{(2)}(\chi)\right) + \left(\check{g}^{(1)}(\chi) - G\check{g}_{\nu}^{(1)}(\chi)\right) v \\
&+ \left(\vec{f}^{(3)}(\chi) - \check{g}^{(1)}(\chi)\, \check{f}_{\nu}^{(2)}(\chi)\right) + \left(\vec{g}^{(2)}(\chi) - \check{g}^{(1)}(\chi)\, \check{g}_{\nu}^{(1)}(\chi)\right) v + O^{(4+)}
\end{aligned}
$$

which ensures that the bottom row of both quadratic order terms are zero. Finally, by applying the "Elimination of g(x)" theorem, we can always force the quadratic order control term to zero, which yields our equation in quadratic normal form

$$
\begin{aligned}
\dot{\chi} &= F\chi + Gv && \text{(V.144)} \\
&+ \left(\check{f}^{(2)}(\chi) - G\check{f}_{\nu}^{(2)}(\chi)\right) \\
&+ \left(\vec{f}^{(3)}(\chi) - \check{g}^{(1)}(\chi)\, \check{f}_{\nu}^{(2)}(\chi)\right) + \left(\vec{g}^{(2)}(\chi) - \check{g}^{(1)}(\chi)\, \check{g}_{\nu}^{(1)}(\chi)\right) v + O^{(4+)}
\end{aligned}
$$

Equation V.144 can be rewritten to put the quadratic order terms into vector/matrix form and to recognize the natural separation between the linearly controllable and linearly uncontrollable states of our system. We get

$$
\begin{aligned}
\begin{bmatrix} \dot{\sigma} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} F_{\sigma} & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \sigma \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} v && \text{(V.145)} \\
&+ \begin{bmatrix} \check{Q}_{\sigma u} & \check{Q}_{\sigma m} & \check{Q}_{\sigma c} \\ \check{Q}_{yu} & \check{Q}_{ym} & \check{Q}_{yc} \end{bmatrix} \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix} + O^{(3+)}
\end{aligned}
$$

where we have neglected the cubic order terms for clarity. The task of this section is to show the form the remaining quadratic order terms take when as many coefficients of the matrix $\check{Q}$ as possible have been eliminated by proper choice of $h^{(2)}(\chi)$.

1. Normal Form of the y Dynamic Equation

Separating out the y dynamics from equation V.145, we have

$$\dot{y} = Ay + Bv + \begin{bmatrix} \check{Q}_{yu} & \check{Q}_{ym} & \check{Q}_{yc} \end{bmatrix} \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix} + O^{(3+)} \qquad \text{(V.146)}$$

Our task is to determine the submatrices $\check{Q}_{yu}$, $\check{Q}_{ym}$ and $\check{Q}_{yc}$ with the fewest possible non-zero coefficients. From the "Feedback Relieves Constraints" theorem and its corollary we can find solutions from equations V.123, V.124 and V.125, while applying the constraints imposed by equations V.129 and V.130, all of which we restate here for convenience.

Equations to solve:

$$\begin{bmatrix} D_c^T & -I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & D_c^T & -I \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} H_{yc_1}^T \\ \vdots \\ \vdots \\ H_{yc_{p-1}}^T \\ H_{yc_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{yc_1}^T - \check{Q}_{yc_1}^T \\ \vdots \\ \vdots \\ \tilde{Q}_{yc_{p-1}}^T - \check{Q}_{yc_{p-1}}^T \\ 0 \end{bmatrix} \qquad \text{(V.147)}$$

$$\begin{bmatrix} D_m^T & -I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & D_m^T & -I \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} H_{ym_1}^T \\ \vdots \\ \vdots \\ H_{ym_{p-1}}^T \\ H_{ym_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{ym_1}^T - \check{Q}_{ym_1}^T \\ \vdots \\ \vdots \\ \tilde{Q}_{ym_{p-1}}^T - \check{Q}_{ym_{p-1}}^T \\ 0 \end{bmatrix} \qquad \text{(V.148)}$$

and

$$\begin{bmatrix} D_\sigma^T & -I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & D_\sigma^T & -I \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} H_{yu_1}^T \\ \vdots \\ \vdots \\ H_{yu_{p-1}}^T \\ H_{yu_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{yu_1}^T - \check{Q}_{yu_1}^T \\ \vdots \\ \vdots \\ \tilde{Q}_{yu_{p-1}}^T - \check{Q}_{yu_{p-1}}^T \\ 0 \end{bmatrix} \tag{V.149}$$

where the matrix $\tilde{Q}$ is defined by the relation

$$\tilde{f}^{(2)}(\chi) = \tilde{Q}\chi^{(2)} = \begin{bmatrix} \tilde{Q}_{\sigma u} & \tilde{Q}_{\sigma m} & \tilde{Q}_{\sigma c} \\ \tilde{Q}_{yu} & \tilde{Q}_{ym} & \tilde{Q}_{yc} \end{bmatrix} \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix} \tag{V.150}$$

Constraints:

$$\begin{bmatrix} D_{B_y}^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & D_{B_y}^T & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} H_{yc_1}^T \\ \vdots \\ H_{yc_{p-1}}^T \\ H_{yc_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{yc_1}^T \\ \vdots \\ \tilde{G}_{yc_{p-1}}^T \\ 0 \end{bmatrix} \tag{V.151}$$

and

$$\begin{bmatrix} D_{B_\sigma}^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & D_{B_\sigma}^T & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} H_{ym_1}^T \\ \vdots \\ H_{ym_{p-1}}^T \\ H_{ym_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{yu_1}^T \\ \vdots \\ \tilde{G}_{yu_{p-1}}^T \\ 0 \end{bmatrix} \tag{V.152}$$

where the matrix $\tilde{G}$ is defined by the relation

$$\tilde{g}^{(1)}(\chi) = \tilde{G}\chi = \begin{bmatrix} \tilde{G}_{\sigma u} & \tilde{G}_{\sigma c} \\ \tilde{G}_{yu} & \tilde{G}_{yc} \end{bmatrix} \begin{bmatrix} \sigma \\ y \end{bmatrix} \tag{V.153}$$

2. Controllable/Controllable Part

The controllable/controllable part is given by the solution

$$\check{Q}_{yc} = \begin{bmatrix} \check{Q}_{yc_1} \\ \vdots \\ \check{Q}_{yc_{p-1}} \\ 0 \end{bmatrix} \qquad \text{(V.154)}$$

to equation V.147, subject to the constraints of equation V.151. We start by stating a theorem due to Kang [Ref. 2].

Theorem C.1 (Controllable/Controllable: Kang's Theorem) *The normal form of equation V.154 is given by*

$$\check{Q}_{yc} y^{(2)} = Q_{y_c} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \qquad \text{(V.155)}$$

where $Q_{y_c} \in R^{p\times p}$ has the upper triangular form (with zeros on the main diagonal and first super-diagonal)

$$Q_{y_c} = \begin{bmatrix} 0 & 0 & \gamma_{13} & \cdots & \gamma_{1p} \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \gamma_{(p-2)p} \\ & & & \ddots & 0 \\ 0 & & \cdots & & 0 \end{bmatrix} \qquad \text{(V.156)}$$

Proof. This theorem is proven in reference [Ref. 10]. ◁

3. Controllable Mixed Part

The controllable/mixed part is given by the solution

$$\check{Q}_{ym} = \begin{bmatrix} \check{Q}_{ym_1} \\ \vdots \\ \check{Q}_{ym_{p-1}} \\ 0 \end{bmatrix} \qquad \text{(V.157)}$$

to equation V.148, subject to the constraints of equation V.152. We state the result in a theorem.

Theorem C.2 (Controllable/Mixed) *The normal form of equation V.157 is given by*

$$\check{Q}_{ym} = 0 \tag{V.158}$$

Proof. We assume the theorem is true and show that we can always find a transformation matrix H_{cm} which satisfies it. Plugging $\check{Q}_{ym} = 0$ into V.148, and eliminating the zero rows, we get

$$\begin{bmatrix} D_m^T & -I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_m^T & -I \end{bmatrix} \begin{bmatrix} H_{ym_1}^T \\ \vdots \\ \vdots \\ H_{ym_{p-1}}^T \\ H_{ym_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{ym_1}^T \\ \vdots \\ \vdots \\ \tilde{Q}_{ym_{p-1}}^T \end{bmatrix} \tag{V.159}$$

which is constrained by equation V.152, from which we eliminate the zero rows and write as

$$\begin{bmatrix} D_{B_\sigma}^T & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & D_{B_\sigma}^T & 0 \end{bmatrix} \begin{bmatrix} H_{ym_1}^T \\ \vdots \\ H_{ym_{p-1}}^T \\ H_{ym_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{yu_1}^T \\ \vdots \\ \tilde{G}_{yu_{p-1}}^T \end{bmatrix} \tag{V.160}$$

We begin by considering that, taken alone, equation V.159 is equivalent to solving $p^2s - ps$ simultaneous equations in the p^2s unknown coefficients of H_{ym}. Adding in the constraints of equation V.160 is equivalent to an additional $ps - s$ simultaneous equations to solve. So, taken together, we have $p^2s - s$ equations in the p^2s coefficients of H_{ym}, which means that our system is underdetermined. This is acceptable, since it only means that more than one transformation matrix H_{ym} may be capable of eliminating all coefficients of $\tilde{Q}_{ym}$. However, we still have to determine whether the

appended matrix $\Delta_{appended} \in R^{(p^2-1)s \times p^2 s}$

$$\Delta_{appended} = \begin{bmatrix} D_m^T & -I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_m^T & -I \\ D_{B_\sigma}^T & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & D_{B_\sigma}^T & 0 \end{bmatrix}$$

contains an invertible $(p^2-1)\,s \times (p^2-1)\,s$ matrix inside it. From Appendix A we have $D_m \in R^{ps \times ps}$ and $D_{B_\sigma} \in R^{ps \times s}$ which, when transposed, yield the block forms

$$D_m^T = \begin{bmatrix} F_\sigma^T & 0 & \cdots & \cdots & 0 \\ I & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I & F_\sigma^T \end{bmatrix} \qquad \text{(V.161)}$$

and

$$D_{B_\sigma}^T = \begin{bmatrix} 0 & \cdots & 0 & I \end{bmatrix} \qquad \text{(V.162)}$$

where $I \in R^{s \times s}$ is the identity matrix in both cases. Using Gaussian elimination [Ref. 12] we put the matrix $\Delta_{appended}$ into upper triangular form. First, perform row exchanges so that each block row of $\Delta_{appended}$ which contains the matrix $D_{B_\sigma}^T$ is stacked directly below the corresponding block row containing the matrix D_m^T. Then, perform row exchanges on $\Delta_{appended}$ to remove all top block rows of D_m^T and block stack them in reverse order on the bottom of the matrix. When row reduction is applied to this matrix, it becomes upper triangular with 1's on the main diagonal, and the first $p^2 s - s$ columns are seen to be independent. This proves that all coefficients of $\tilde{Q}_{ym}$ can be eliminated, and proves the theorem. ◃

4. Controllable/Uncontrollable Part

The controllable/uncontrollable part is given by the solution

$$\check{Q}_{yu} = \begin{bmatrix} \check{Q}_{yu_1} \\ \vdots \\ \check{Q}_{yu_{p-1}} \\ 0 \end{bmatrix} \tag{V.163}$$

to equation V.149. This part is not subject to any constraint equations. We start by stating a theorem.

Theorem C.3 (Controllable/Uncontrollable) *The normal form of equation V.163 is given by*

$$\check{Q}_{yu} = 0 \tag{V.164}$$

Proof. We will prove the theorem constructively, that is we will assume it is true, and then find the transformation which satisfies it. Plugging $\check{Q}_{yu} = 0$ into V.149, and eliminating the zero rows, we get

$$\begin{bmatrix} D_\sigma^T & -I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & D_\sigma^T & -I \end{bmatrix} \begin{bmatrix} H_{yu_1}^T \\ \vdots \\ \vdots \\ H_{yu_{p-1}}^T \\ H_{yu_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{yu_1}^T \\ \vdots \\ \vdots \\ \tilde{Q}_{yu_{p-1}}^T \end{bmatrix} \tag{V.165}$$

which can be solved regardless of the value of D_σ^T as

$$H_{yu_1}^T = \textit{arbitrary} \tag{V.166}$$

$$H_{yu_2}^T = D_\sigma^T H_{yu_1}^T - \tilde{Q}_{yu_1}^T \tag{V.167}$$

$$\vdots \tag{V.168}$$

$$H_{yu_p}^T = D_\sigma^T H_{yu_{p-1}}^T - \tilde{Q}_{yu_{p-1}}^T \tag{V.169}$$

In the actual solution, we will pick $H_{yu_1}^T = 0$. $\triangleleft$

5. Normal Form of the σ Dynamic Equation

Separating out the σ dynamics from equation V.145, we have

$$\dot{\sigma} = F_\sigma \sigma + \begin{bmatrix} \check{Q}_{\sigma u} & \check{Q}_{\sigma m} & \check{Q}_{\sigma c} \end{bmatrix} \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix} + O^{(3+)} \tag{V.170}$$

where we want to determine the submatrices $\check{Q}_{\sigma u}$, $\check{Q}_{\sigma m}$ and $\check{Q}_{\sigma c}$ with the fewest possible non-zero coefficients. We can find solutions for these submatrices from equations V.104, V.105 and V.106, which we restate here for convenience

$$H_{\sigma c} D_c - F_\sigma H_{\sigma c} = \tilde{Q}_{\sigma c} - \check{Q}_{\sigma c} \tag{V.171}$$

$$H_{\sigma m} D_m - F_\sigma H_{\sigma m} = \tilde{Q}_{\sigma m} - \check{Q}_{\sigma m} \tag{V.172}$$

$$H_{\sigma u} D_\sigma - F_\sigma H_{\sigma u} = \tilde{Q}_{\sigma u} - \check{Q}_{\sigma u} \tag{V.173}$$

The constraints are found from equations V.109 and V.110, which we also restate

$$\tilde{G}_{\sigma u} - H_{\sigma m} D_{B_\sigma} = 0 \tag{V.174}$$

$$\tilde{G}_{\sigma c} - H_{\sigma c} D_{B_y} = 0 \tag{V.175}$$

Equations V.171, V.172 and V.173 can be put into solvable form by applying the Unstacking Theorem, and likewise for equations V.174 and V.175 by applying the Unstacking Corollary. We state these here.

Equations to solve:

$$\left(\begin{bmatrix} D_c^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_c^T \end{bmatrix} - \begin{bmatrix} F_{\sigma_{11}} I & \cdots & \cdots & F_{\sigma_{1s}} I \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ F_{\sigma_{s1}} I & \cdots & \cdots & F_{\sigma_{ss}} I \end{bmatrix} \right) \begin{bmatrix} H_{\sigma c_1}^T \\ \vdots \\ \vdots \\ H_{\sigma c_s}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{\sigma c_1}^T - \check{Q}_{\sigma c_1}^T \\ \vdots \\ \vdots \\ \tilde{Q}_{\sigma c_s}^T - \check{Q}_{\sigma c_s}^T \end{bmatrix} \tag{V.176}$$

$$\left(\begin{bmatrix} D_m^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_m^T \end{bmatrix} - \begin{bmatrix} F_{\sigma_{11}} I & \cdots & \cdots & F_{\sigma_{1s}} I \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ F_{\sigma_{s1}} I & \cdots & \cdots & F_{\sigma_{ss}} I \end{bmatrix} \right) \begin{bmatrix} H_{\sigma m_1}^T \\ \vdots \\ \vdots \\ H_{\sigma m_s}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{\sigma m_1}^T - \check{Q}_{\sigma m_1}^T \\ \vdots \\ \vdots \\ \tilde{Q}_{\sigma m_s}^T - \check{Q}_{\sigma m_s}^T \end{bmatrix} \tag{V.177}$$

and

$$\left(\begin{bmatrix} D_\sigma^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_\sigma^T \end{bmatrix} - \begin{bmatrix} F_{\sigma_{11}} I & \cdots & \cdots & F_{\sigma_{1s}} I \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ F_{\sigma_{s1}} I & \cdots & \cdots & F_{\sigma_{ss}} I \end{bmatrix}\right) \begin{bmatrix} H_{\sigma u_1}^T \\ \vdots \\ \vdots \\ H_{\sigma u_s}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{\sigma u_1}^T - \check{Q}_{\sigma u_1}^T \\ \vdots \\ \vdots \\ \tilde{Q}_{\sigma u_s}^T - \check{Q}_{\sigma u_s}^T \end{bmatrix} \tag{V.178}$$

where the matrix $\tilde{Q}$ is defined by the block relation

$$\tilde{f}^{(2)}(\chi) = \tilde{Q}\chi^{(2)} = \begin{bmatrix} \tilde{Q}_{\sigma u} & \tilde{Q}_{\sigma m} & \tilde{Q}_{\sigma c} \\ \tilde{Q}_{yu} & \tilde{Q}_{ym} & \tilde{Q}_{yc} \end{bmatrix} \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix} \tag{V.179}$$

Constraints:

$$\begin{bmatrix} D_{B_y}^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_{B_y}^T \end{bmatrix} \begin{bmatrix} H_{\sigma c_1}^T \\ \vdots \\ \vdots \\ H_{\sigma c_s}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{\sigma c_1}^T \\ \vdots \\ \vdots \\ \tilde{G}_{\sigma c_s}^T \end{bmatrix} \tag{V.180}$$

and

$$\begin{bmatrix} D_{B_\sigma}^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_{B_\sigma}^T \end{bmatrix} \begin{bmatrix} H_{\sigma m_1}^T \\ \vdots \\ \vdots \\ H_{\sigma m_s}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{\sigma u_1}^T \\ \vdots \\ \vdots \\ \tilde{G}_{\sigma u_s}^T \end{bmatrix} \tag{V.181}$$

where the matrix $\tilde{G}$ is defined by the block relation

$$\tilde{g}^{(1)}(\chi) = \tilde{G}\chi = \begin{bmatrix} \tilde{G}_{\sigma u} & \tilde{G}_{\sigma c} \\ \tilde{G}_{yu} & \tilde{G}_{yc} \end{bmatrix} \begin{bmatrix} \sigma \\ y \end{bmatrix} \tag{V.182}$$

Before we begin considering individual cases, we need to prove two useful lemmas and a useful theorem.

Lemma C.4 (Inverse of an Upper Triangular Matrix) *The inverse of an invertible upper triangular matrix is upper triangular and invertible.*

Proof. This is a well known result from linear algebra. Let the invertible upper triangular matrix $\beta \in R^{h\times h}$ be given by

$$\beta = \begin{bmatrix} \beta_{11} & \cdots & \cdots & \beta_{1h} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \beta_{hh} \end{bmatrix} \tag{V.183}$$

such that β^{-1} exists, that is such that $\beta_{ii} \neq 0$ for $i = 1$ to h. Let $\alpha = \beta^{-1}$ be given by

$$\alpha = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1h} \\ \vdots & & \vdots \\ \alpha_{h1} & \cdots & \alpha_{hh} \end{bmatrix} \tag{V.184}$$

In block form we write

$$\beta = \begin{bmatrix} \beta_{11} & \bar{\beta}_1 \\ 0 & \hat{\beta} \end{bmatrix} \tag{V.185}$$

and

$$\alpha = \begin{bmatrix} \alpha_{11} & \bar{\alpha}_1 \\ \tilde{\alpha}_1 & \hat{\alpha} \end{bmatrix} \tag{V.186}$$

Since $\alpha\beta = I$, we have

$$\begin{bmatrix} \alpha_{11} & \bar{\alpha}_1 \\ \tilde{\alpha}_1 & \hat{\alpha} \end{bmatrix} \begin{bmatrix} \beta_{11} & \bar{\beta}_1 \\ 0 & \hat{\beta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}$$

which yields

$$\alpha_{11} = \frac{1}{\beta_{11}} \tag{V.187}$$

$$\tilde{\alpha}_1 = 0 \tag{V.188}$$

$$\bar{\alpha}_1 = -\frac{1}{\beta_{11}}\bar{\beta}_1\hat{\beta}^{-1} \tag{V.189}$$

$$\hat{\alpha} = \hat{\beta}^{-1} \tag{V.190}$$

where we have used the fact that $\beta_{11} \neq 0$ to obtain $\tilde{\alpha}_1 = 0$. Since $\tilde{\alpha}_1$ is the vector of all elements directly below α_{11}, the matrix α has all zeros below the main diagonal. By

induction, this process can be repeated using $\hat{\beta}$ and $\hat{\alpha}$ in place of β and α. Therefore α is upper triangular, and since β is invertible, α is also invertible. This proves the lemma. ◁

Lemma C.5 (Zero Diagonal Upper Triangular Matrix) *Given a matrix of the form*

$$F_\delta = F_\alpha F_\gamma F_\beta \tag{V.191}$$

where $F_\alpha \in R^{k\times k}$, $F_\beta \in R^{k\times k}$ and $F_\gamma \in R^{k\times k}$ are upper triangular matrices, with one of them having zeroes on the main diagonal, then the matrix F_δ is upper triangular with zeroes on the main diagonal.

Proof. This is another well known result from linear algebra. We are given the structure of the matrices F_α, F_β and F_γ, which we write in block form as

$$F_\alpha = \begin{bmatrix} F_{\alpha 11} & F_{\alpha 12} & \cdots & F_{\alpha 1k} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & F_{\alpha k-1k} \\ 0 & \cdots & 0 & F_{\alpha kk} \end{bmatrix} = \begin{bmatrix} F_{\alpha 11} & \bar{F}_{\alpha 1} \\ 0 & \hat{F}_\alpha \end{bmatrix} \tag{V.192}$$

$$F_\beta = \begin{bmatrix} F_{\beta 11} & F_{\beta 12} & \cdots & F_{\beta 1k} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & F_{\beta k-1k} \\ 0 & \cdots & 0 & F_{\beta kk} \end{bmatrix} = \begin{bmatrix} F_{\beta 11} & \bar{F}_{\beta 1} \\ 0 & \hat{F}_\beta \end{bmatrix} \tag{V.193}$$

$$F_\gamma = \begin{bmatrix} F_{\gamma 11} & \cdots & \cdots & F_{\gamma 1k} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & F_{\gamma kk} \end{bmatrix} = \begin{bmatrix} F_{\gamma 11} & \bar{F}_{\gamma 1} \\ 0 & \hat{F}_\gamma \end{bmatrix} \tag{V.194}$$

where the upper left block is a scalar in each case. Multiplying out by blocks, we get

$$\begin{aligned} F_\delta &= \begin{bmatrix} F_{\alpha 11} & \bar{F}_{\alpha 1} \\ 0 & \hat{F}_\alpha \end{bmatrix} \begin{bmatrix} F_{\gamma 11} & \bar{F}_{\gamma 1} \\ 0 & \hat{F}_\gamma \end{bmatrix} \begin{bmatrix} F_{\beta 11} & \bar{F}_{\beta 1} \\ 0 & \hat{F}_\beta \end{bmatrix} \\ &= \begin{bmatrix} F_{\alpha 11}F_{\gamma 11}F_{\beta 11} & F_{\alpha 11}F_{\gamma 11}\bar{F}_{\beta 1} + F_{\alpha 11}\bar{F}_{\gamma 1}\hat{F}_\beta + \bar{F}_{\alpha 1}\hat{F}_\gamma\hat{F}_\beta \\ 0 & \hat{F}_\alpha\hat{F}_\gamma\hat{F}_\beta \end{bmatrix} \end{aligned} \tag{V.195}$$

which is seen, by induction, to be upper triangular with zeroes on the main diagonal, since all elements below the diagonal are zero, and since the scalar diagonal element is zero if the diagonal elements of F_α, F_β or F_γ are zero. ◁

Theorem C.6 (Invertible Matrix) *Given a matrix $\Delta \in R^{kh \times kh}$ of the block form*

$$\Delta = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I \end{bmatrix} + \begin{bmatrix} F_{11} & \cdots & \cdots & F_{1k} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ F_{k1} & \cdots & \cdots & F_{kk} \end{bmatrix} \tag{V.196}$$

where $I \in R^{h \times h}$ is the identity matrix, and the submatrices $F_{ij} \in R^{h \times h}$ are upper triangular with zeroes on the main diagonal, then the matrix Δ is invertible.

Proof. We will show that Δ is an invertible matrix by putting it into upper triangular form through block Gaussian elimination and induction. First, if $k = 1$, then $\Delta = I + F_{11}$, which is upper triangular with non-zero elements on the main diagonal, and is therefore invertible. If $k > 1$, then Δ can be represented in block form as

$$\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} \tag{V.197}$$

where

$$\Delta_{11} = I + F_{11} \tag{V.198}$$

$$\Delta_{12} = \begin{bmatrix} F_{12} & \cdots & F_{1k} \end{bmatrix} \tag{V.199}$$

$$\Delta_{21} = \begin{bmatrix} F_{21} \\ \vdots \\ F_{k1} \end{bmatrix} \tag{V.200}$$

$$\Delta_{22} = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I \end{bmatrix} + \begin{bmatrix} F_{22} & \cdots & \cdots & F_{2k} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ F_{k2} & \cdots & \cdots & F_{kk} \end{bmatrix} \tag{V.201}$$

and we note that Δ_{11} is an upper triangular invertible matrix. Now, using block Gaussian elimination on equation V.197 gives

$$\begin{aligned}\hat{M}_1\Delta &= \begin{bmatrix} I & 0 \\ M_1 & I \end{bmatrix}\begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} \\ &= \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ M_1\Delta_{11} + \Delta_{21} & M_1\Delta_{12} + \Delta_{22} \end{bmatrix}\end{aligned} \tag{V.202}$$

Since Δ_{11} is invertible, we can choose $M_1 = -\Delta_{21}\Delta_{11}^{-1}$, which gives

$$\hat{M}_1\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ 0 & \Delta_{22} - \Delta_{21}\Delta_{11}^{-1}\Delta_{12} \end{bmatrix} \tag{V.203}$$

which is block upper triangular. Now, looking at the term $\Delta_{22} - \Delta_{21}\Delta_{11}^{-1}\Delta_{12}$, we have

$$\begin{aligned}&\Delta_{22} - \Delta_{21}\Delta_{11}^{-1}\Delta_{12} \\ &= \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I \end{bmatrix} + \begin{bmatrix} F_{22} & \cdots & \cdots & F_{2k} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ F_{k2} & \cdots & \cdots & F_{kk} \end{bmatrix} - \begin{bmatrix} F_{21} \\ \vdots \\ F_{k1} \end{bmatrix} [I + F_{11}]^{-1} \begin{bmatrix} F_{12} & \cdots & F_{1k} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I \end{bmatrix} + \begin{bmatrix} F_{22} - F_{21}[I + F_{11}]^{-1}F_{12} & \cdots & F_{2k} - F_{21}[I + F_{11}]^{-1}F_{1k} \\ \vdots & & \vdots \\ F_{k2} - F_{k1}[I + F_{11}]^{-1}F_{12} & \cdots & F_{kk} - F_{k1}[I + F_{11}]^{-1}F_{1k} \end{bmatrix}\end{aligned} \tag{V.204}$$

Now look at the block terms in the second matrix. By the "Inverse of an Upper Triangular Matrix" lemma, the term $[I + F_{11}]^{-1}$ is an upper triangular invertible matrix. Using that result, the combinations of the form $F_{i1}[I + F_{11}]^{-1}F_{1j}$ are all upper triangular matrices with zeroes on the main diagonal by the "Zero Diagonal Upper Triangular Matrix" lemma. We are given that the matrices F_{ij} are upper triangular with zeroes on the main diagonal, and a sum (or difference) of two matrices of this type is also an upper triangular matrix with zeroes on the main diagonal. So, we are back to our original problem, only one block smaller. Repeating this

process through induction yields a block upper triangular matrix such that all blocks on the main diagonal are upper triangular invertible matrices. Therefore, Gaussian elimination can convert our original matrix Δ to upper triangular form with non-zeroes on the main diagonal, so Δ is invertible, which proves the theorem. $\triangleleft$

Now that we've established some preliminary results, we can consider the individual cases.

6. Uncontrollable/Controllable Part

The uncontrollable/controllable part of the normal form is given by the solution

$$\check{Q}_{\sigma c} = \begin{bmatrix} \check{Q}_{\sigma c_1} \\ \vdots \\ \check{Q}_{\sigma c_p} \end{bmatrix} \tag{V.205}$$

to equation V.176, subject to the constraints of equation V.180. We state our results in a theorem.

Theorem C.7 (Uncontrollable/Controllable) *The normal form of equation V.205 is given by*

$$\check{Q}_{\sigma c} y^{(2)} = Q_{\sigma_c} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \tag{V.206}$$

where $Q_{\sigma_c} \in R^{s \times p}$ has the block form

$$Q_{\sigma_c} = \begin{bmatrix} 0 \\ Q_{z_c} \\ Q_{w_c} \end{bmatrix} \tag{V.207}$$

with $Q_{z_c} \in R^{q \times p}$ and $Q_{w_c} \in R^{m \times p}$.

Proof. We are trying to solve equation V.176 subject to the constraints of equation V.180. Equation V.176 has the form of $\frac{sp(p+1)}{2}$ simultaneous equations in $\frac{sp(p+1)}{2}$ unknown coefficients of $H_{\sigma c}$, since each row of the transformation matrix $H_{\sigma c}$

is $H_{\sigma c_i}^T \in R^{\frac{p(p+1)}{2}}$, with $i = 1$ to s. From Appendix A we have $D_{B_y} \in R^{\frac{p(p+1)}{2} \times p}$ which, when transposed, yields the block form

$$D_{B_y}^T = \begin{bmatrix} 0 & I_2 \end{bmatrix} \tag{V.208}$$

where $I_2 \in R^{p \times p}$ is given in block form by

$$I_2 = \begin{bmatrix} I & 0 \\ 0 & 2 \end{bmatrix} \tag{V.209}$$

where the bottom right element is the scalar 2. When we apply the constraints of equation V.180, we see that each matrix D_{B_y} imposes p constraints, and there are s matrices D_{B_y} involved, for a total of ps constraints. So, equation V.176 is overdetermined by at least ps terms when the constraints are imposed, which means we will have at least ps terms which we can't eliminate in the uncontrollable/controllable part of our normal form. Although different normal forms are possible, there is a natural choice in this case, which is to pick the set of all coefficients of terms of the form y_i^2, for $i = 1$ to p, which we will now investigate. Equation V.176 has s potentially resonant terms (with actual resonance depending on the coefficients of the matrix F_σ, which is arbirary), where a resonant term is defined as a term in the matrix $Q_{\sigma c}$ which cannot be affected at all by our choice of the transformation matrix $H_{\sigma c}$. The only way to have a resonant term in equation V.177 is to have a zero row in the matrix Δ_c, where Δ_c is the matrix

$$\Delta_c = \begin{bmatrix} D_c^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_c^T \end{bmatrix} - \begin{bmatrix} F_{\sigma_{11}} I & \cdots & \cdots & F_{\sigma_{1s}} I \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ F_{\sigma_{s1}} I & \cdots & \cdots & F_{\sigma_{ss}} I \end{bmatrix} \tag{V.210}$$

Given that the submatrices $F_{\sigma_{ij}} I \in R^{p\frac{p+1}{2} \times p\frac{p+1}{2}}$ in the second matrix are diagonal, a zero row in Δ_c can only occur in those rows of the matrix D_c^T in which all the off-diagonal elements are zero. From Appendix A we have an iterative definition for

$D_{c_{p+1}} \in R^{\frac{(p+1)(p+2)}{2} \times \frac{(p+1)(p+2)}{2}}$ in terms of $D_{c_p} \in R^{\frac{p(p+1)}{2} \times \frac{p(p+1)}{2}}$ which, when transposed, yields the block form

$$D^T_{c_{p+1}} = \begin{bmatrix} D^T_{c_p} & 0 & 0 \\ D^T_{B_y} & A^T & 0 \\ 0 & B^T & 0 \end{bmatrix} \tag{V.211}$$

where $D^T_{B_y}$ was given above, $A^T \in R^{p \times p}$ is given by

$$A^T = \begin{bmatrix} 0 & & \cdots & & 0 \\ 1 & \ddots & & & \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \tag{V.212}$$

and $B^T \in R^{1 \times p}$ given by

$$B^T = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \tag{V.213}$$

So, we see that there is no possibility of a zero row in the two bottom block rows of $D^T_{c_{p+1}}$ since all rows in $D^T_{B_y}$ and B^T are non-zero. So we check the top block row, which means we are checking $D^T_{c_p}$. By induction, there is no possibility of a zero row until we reach matrices D^T_c which we have actually calculated out. We have

$$D^T_{c_2} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{V.214}$$

and we can see that there is a zero row, and therefore a potential resonant term, on the top row of D^T_c. This corresponds to the coefficient of the y_1^2 term, and we see that our normal form set, consisting of all terms y_i^2 includes the potential resonance as a member. Our choice of normal form set consisting of all terms y_i^2 also guarantees that the unconstrained coefficients of the transformation matrix $H_{\sigma m}$ are able to exactly eliminate all remaining coefficients of $\tilde{Q}_{\sigma m}$, as we now show. Choosing the normal form set is equivalent to removing the rows corresponding to those coefficients,

as detailed in Appendix B. Applying the constraints imposed by equation V.180 is equivalent to removing the columns of the matrix Δ_c corresponding to the coefficients of the $y_i y_p$ terms in the matrix $H_{\sigma c}$, which is equivalent to removing those columns of Δ_c containing the two far right block column in any of the matrices D_c^T, as detailed in Appendix B. Truncating the matrix D_c^T in this fashion (removing p appropriate rows and p appropriate columns) results in a matrix which is upper triangular and invertible, as we show by induction. Start with a matrix $D_{c_p}^T$ which, after truncation, is upper triangular invertible. (We start with $p = 2$, and we truncate the matrix $D_{c_2}^T$ by removing the first row (y_1^2 terms) and third row (y_2^2 terms), and second and third columns (constraints on H_{yc}). The result is

$$D_{c_{2trunc}}^T = [2] \tag{V.215}$$

which is an upper triangular, invertible matrix. This gives us a starting place.) When the matrix $D_{c_p}^T$ is appended to create $D_{c_{p+1}}^T$, the result is upper triangular invertible when it is truncated. We see this by noticing that the matrix $D_{c_{p+1}}^T$ is given in equation V.211, and when it is truncated the two rightmost block columns are removed (constraints on H_{yc}), the bottom row containing B^T is removed (y_{p+1}^2 terms), and all the previous rows which were removed from $D_{c_p}^T$ (y_1^2 through y_p^2 terms) are also removed. (Note that the blocks in $D_{c_p}^T$ which were removed as columns are not removed in $D_{c_{p+1}}^T$, but that they do not effect our argument, since they are above and to the right of the main diagonal of the truncated matrix.) Now, after the truncation, the non-zero terms in $D_{B_y}^T$ line up on the diagonal of the new truncated matrix, and connect with the diagonal terms from the truncated $D_{c_p}^T$ block. So the new appended matrix $D_{c_p+1}^T$ is upper triangular invertible when it is truncated, and the process can be repeated indefinitely. This completes the proof by induction, and shows that the matrix D_c^T is upper triangular and invertible after it has been truncated. The end result is that this truncation process results in a new matrix Δ_{ctrunc}, which is

invertible, given by

$$\Delta_{ctrunc} = \begin{bmatrix} D^T_{C_{trunc}} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D^T_{C_{trunc}} \end{bmatrix} - \begin{bmatrix} (F_{\sigma_{11}} I)_{trunc} & \cdots & \cdots & (F_{\sigma_{1s}} I)_{trunc} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ (F_{\sigma_{s1}} I)_{trunc} & \cdots & \cdots & (F_{\sigma_{ss}} I)_{trunc} \end{bmatrix} \tag{V.216}$$

That Δ_{ctrunc} is invertible can be seen by noting that the truncation process moved the elements of the matrices $F_{\sigma_{ij}} I$ off the diagonal up and to the right, so that $\left(F_{\sigma_{ij}} I\right)_{trunc}$ is upper triangular with zeroes on the main diagonal, and then applying the "Invertible Matrix" theorem, since $D^T_{C_{trunc}}$ is upper triangular and invertible. Since there must be at least ps elements in the normal form, and since picking the coefficients of y_i^2 as the normal form set results in no additional outside elements possible in the normal form, the coefficients of y_i^2 are an acceptable normal form. (Other normal forms may also be possible, but they are not of concern to us.) Noting that we can block separate out the μ, z and w components, and that the μ components are zero (since that's what we started with) completes the proof of the theorem. ◁

7. Uncontrollable/Mixed Part

The controllable/mixed part is given by the solution

$$\check{Q}_{\sigma m} = \begin{bmatrix} \check{Q}_{\sigma m_1} \\ \vdots \\ \check{Q}_{\sigma m_p} \end{bmatrix} \tag{V.217}$$

to equation V.177, subject to the constraints of equation V.181. We state our result in a theorem.

Theorem C.8 (Uncontrollable/Mixed) *The normal form of equation V.217 is given by*

$$\check{Q}_{\sigma m} \sigma y^{(2)} = Q_{\sigma_m} [\sigma y_1] \tag{V.218}$$

where the notation $[\sigma y_1]$ indicates the vector σ multiplied by the scalar first component of the vector y. Additionally, the matrix $Q_{\sigma_m} \in R^{s \times s}$ has the block form

$$Q_{\sigma_m} = \begin{bmatrix} 0 & 0 \\ Q_{z_{m_1}} & Q_{z_{m_2}} \\ Q_{w_{m_1}} & Q_{w_{m_2}} \end{bmatrix} \tag{V.219}$$

with $Q_{z_{m_1}} \in R^{q \times (r+q)}$, $Q_{z_{m_2}} \in R^{q \times m}$, $Q_{w_{m_1}} \in R^{m \times (r+q)}$ and $Q_{w_{m_2}} \in R^{m \times m}$.

Proof. We are trying to solve equation V.177 subject to the constraints of equation V.181. Equation V.177 has the form of s^2p simultaneous equations in s^2p unknowns, since each row of the transformation matrix $H_{\sigma m}$ is $H^T_{\sigma m_i} \in R^{ps}$, with $i = 1$ to s. From Appendix A we have $D_{B_\sigma} \in R^{ps \times s}$ which, when transposed, yields the block form

$$D^T_{B_\sigma} = \begin{bmatrix} 0 & \cdots & 0 & I \end{bmatrix} \tag{V.220}$$

where $I \in R^{s \times s}$ is the identity matrix. When we apply the constraints of equation V.181, we see that each matrix D_{B_σ} imposes s constraints, and there are s matrices D_{B_σ} involved, for a total of s^2 constraints. So, equation V.177 is overdetermined by at least s^2 terms when the constraints are imposed, which means we will have at least s^2 terms which we can't eliminate in the uncontrollable/mixed part of our normal form. Although different normal forms are possible, there is a natural choice in this case, which is to pick the set of all coefficients of terms of the form $\sigma_i y_i$ for $i = 1$ to s, which we will now investigate. Equation V.177 has s^2 potentially resonant terms (with actual resonance depending on the specific coefficients of the matrix F_σ, which are still undetermined), where a resonant term is defined as a term in the matrix $Q_{\sigma m}$ which cannot be affected by our choice of the transformation matrix $H_{\sigma m}$. The only way to have a resonant term in equation V.177 is to have a zero row in the matrix Δ_m, where Δ_m is the matrix

$$\Delta_m = \begin{bmatrix} D^T_m & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D^T_m \end{bmatrix} - \begin{bmatrix} F_{\sigma_{11}} I & \cdots & \cdots & F_{\sigma_{1s}} I \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ F_{\sigma_{s1}} I & \cdots & \cdots & F_{\sigma_{ss}} I \end{bmatrix} \tag{V.221}$$

Given that the submatrices $F_{\sigma_{ij}} I \in R^{ps \times ps}$ in the second matrix are diagonal, a zero row in Δ_m can only occur in those rows of the matrix D_m^T in which all the off-diagonal elements are zero. From Appendix A we have $D_m \in R^{ps \times ps}$ which, when transposed, yields the block form

$$D_m^T = \begin{bmatrix} F_\sigma^T & 0 & \cdots & \cdots & 0 \\ I & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I & F_\sigma^T \end{bmatrix} \tag{V.222}$$

where $I \in R^{s \times s}$ is the identity matrix. Now, examining the structure of D_m^T, potential zero rows can only occur in the top block row, since the identity matrix blocks are off-diagonal in all the other rows. Since there are s potential zero rows for each matrix D_m^T, and s of these matrices in the matrix Δ_m, there are a total of s^2 potentially resonant terms, as previously stated. The set of potentially resonant terms consists of all the coefficients of the quadratic state vector component σy_1, as can be seen by the fact that the top s rows in each matrix D_m^T correspond to the first s elements of a corresponding row in the matrices $\tilde{Q}_{\sigma m}$, $H_{\sigma m}$, etc., which are the coefficients of the σy_1 terms. Now, if we choose this set of potentially resonant terms as our normal form, then the unconstrained coefficients of the transformation matrix $H_{\sigma m}$ are guaranteed to be able to exactly eliminate all remaining coefficients of $\tilde{Q}_{\sigma m}$, as we now show. Choosing the set of potentially resonant terms as our normal form is equivalent to removing their rows, as detailed in Appendix B. Applying the constraints imposed by equation V.181 is equivalent to removing the columns of the matrix Δ_m corresponding to the coefficients of the σy_p terms in the matrix $H_{\sigma m}$, which is equivalent to removing those columns of Δ_m containing a far right block column in D_m^T, as detailed in Appendix B. The end result is that this truncation process results in a new matrix Δ_{mtrunc}, which is invertible, given by

$$\Delta_{mtrunc} = \begin{bmatrix} D_{m_{trunc}}^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_{m_{trunc}}^T \end{bmatrix} - \begin{bmatrix} (F_{\sigma_{11}} I)_{trunc} & \cdots & \cdots & (F_{\sigma_{1s}} I)_{trunc} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ (F_{\sigma_{s1}} I)_{trunc} & \cdots & \cdots & (F_{\sigma_{ss}} I)_{trunc} \end{bmatrix} \tag{V.223}$$

with

$$D^T_{m_{trunc}} = \begin{bmatrix} I & F^T_\sigma & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & F^T_\sigma \\ 0 & \cdots & \cdots & 0 & I \end{bmatrix} \tag{V.224}$$

and

$$\left(F_{\sigma_{ij}} I\right)_{trunc} = \begin{bmatrix} 0 & F_{\sigma_{ij}} I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & F_{\sigma_{ij}} I \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \tag{V.225}$$

That Δ_{mtrunc} is invertible can be seen by noticing that $D^T_{m_{trunc}}$ is upper triangular invertible and that $\left(F_{\sigma_{ij}} I\right)_{trunc}$ is upper triangular with zeroes on the main diagonal, and then applying the "Invertible Matrix" theorem. Since there must be at least s^2 elements in the normal form, and since picking the coefficients of σy_1 as the normal form set results in no additional outside elements possible in the normal form, the coefficients of σy_1 are an acceptable normal form. (Other normal forms may also be possible, but they are not of concern to us.) Now we look at the separation of the matrix Q_{σ_m} into blocks. Noting that we can block separate out the μ, z and w components, that the μ components are zero (since that's what we started with), and that we can block separate the vector σy_1 as

$$\sigma y_1 = \begin{bmatrix} \xi y_1 \\ w y_1 \end{bmatrix}$$

completes the proof of the theorem. ◁

8. Uncontrollable/Uncontrollable Part

The uncontrollable/uncontrollable part is given by the solution to the equation V.173, which we restate here as

$$H_{\sigma u} D_\sigma - F_\sigma H_{\sigma u} = \tilde{Q}_{\sigma u} - \check{Q}_{\sigma u} \tag{V.226}$$

and where we wish to solve for $\check{Q}_{\sigma u}$, the normal form for this part. We start by recognizing that, of the six separate pieces of our original homological equation, this is the only piece which is completely unconnected to any of the linearly controllable states. So, solving for the normal form of this piece of the homological equation is equivalent to finding the Poincare normal form of a dynamic equation without control, a job which has a large body of literature to assist us. Although we will not be able to solve equation V.226 for the general case, we will be able to make some statements about its general structure, based on the structure of F_σ and D_σ. We start with a theorem which splits off the influence of the states w from the rest.

Theorem C.9 (Poincare Normal Form, Part I) *When the linearly uncontrollable state matrix F_σ has the block diagonal form*

$$F_\sigma = \begin{bmatrix} F_\xi & 0 \\ 0 & F_w \end{bmatrix} \tag{V.227}$$

and the linearly uncontrollable state vector σ is given in block form by

$$\sigma = \begin{bmatrix} \xi \\ w \end{bmatrix} \tag{V.228}$$

the solutions $\check{Q}_{\sigma u}$ and $H_{\sigma u}$ to equation V.226 are given by

$$\check{Q}_{\sigma u} = \begin{bmatrix} \check{Q}_{\xi\xi} & \check{Q}_{\xi m} & \check{Q}_{\xi w} \\ \check{Q}_{w\xi} & \check{Q}_{wm} & \check{Q}_{ww} \end{bmatrix} \tag{V.229}$$

$$H_{\sigma u} = \begin{bmatrix} H_{\xi\xi} & H_{\xi m} & H_{\xi w} \\ H_{w\xi} & H_{wm} & H_{ww} \end{bmatrix} \tag{V.230}$$

With $\tilde{Q}_{\sigma u}$ given by

$$\tilde{Q}_{\sigma u} = \begin{bmatrix} \tilde{Q}_{\xi\xi} & \tilde{Q}_{\xi m} & \tilde{Q}_{\xi w} \\ \tilde{Q}_{w\xi} & \tilde{Q}_{wm} & \tilde{Q}_{ww} \end{bmatrix} \tag{V.231}$$

the block elements of $\check{Q}_{\sigma u}$ and $H_{\sigma u}$ are found as solutions of the separable matrix equations

$$H_{ww}D_w - F_w H_{ww} = \tilde{Q}_{ww} - \check{Q}_{ww} \tag{V.232}$$

$$H_{wm}D_{u_m} - F_w H_{wm} = \tilde{Q}_{wm} - \check{Q}_{wm} \tag{V.233}$$

$$H_{w\xi}D_\xi - F_w H_{w\xi} = \tilde{Q}_{w\xi} - \check{Q}_{w\xi} \tag{V.234}$$

$$H_{\xi w}D_w - F_\xi H_{\xi w} = \tilde{Q}_{\xi w} - \check{Q}_{\xi w} \tag{V.235}$$

$$H_{\xi m}D_{u_m} - F_\xi H_{\xi m} = \tilde{Q}_{\xi m} - \check{Q}_{\xi m} \tag{V.236}$$

$$H_{\xi\xi}D_\xi - F_\xi H_{\xi\xi} = \tilde{Q}_{\xi\xi} - \check{Q}_{\xi\xi} \tag{V.237}$$

which individually have the form needed to apply the Unstacking Theorem for solution, and where the structural matrices D_w, D_{u_m} and D_ξ are given in Appendix A.

Proof. From Appendix A we have

$$D_\sigma = \begin{bmatrix} D_\xi & 0 & 0 \\ 0 & D_{u_m} & 0 \\ 0 & 0 & D_w \end{bmatrix} \tag{V.238}$$

which we plug into equation V.226 along with equation V.227 and get the desired result by direct calculation. ◃

Now we can look at how to calculate the Poincare normal form of our center states, ξ, when there is no influence from the states w.

Theorem C.10 (Poincare Normal Form, Part II) *The Poincare normal form of the center states*

$$\xi = \begin{bmatrix} \mu \\ z \end{bmatrix} \tag{V.239}$$

when there is no influence from the states w, is found by solving equation V.237 which we rewrite here as

$$H_{\xi\xi}D_\xi - F_\xi H_{\xi\xi} = \tilde{Q}_{\xi\xi} - \check{Q}_{\xi\xi} \tag{V.240}$$

for the coefficient matrices $H_{\xi\xi}$ and $\check{Q}_{\xi\xi}$, which are given by

$$\check{Q}_{\xi\xi} = \begin{bmatrix} 0 & 0 & 0 \\ \check{Q}_{z\mu} & \check{Q}_{zm} & \check{Q}_{zz} \end{bmatrix} \tag{V.241}$$

$$H_{\xi\xi} = \begin{bmatrix} 0 & 0 & 0 \\ H_{z\mu} & H_{zm} & H_{zz} \end{bmatrix} \tag{V.242}$$

With $\tilde{Q}_{\xi\xi}$ and F_ξ given by

$$\tilde{Q}_{\xi\xi} = \begin{bmatrix} 0 & 0 & 0 \\ \tilde{Q}_{z\mu} & \tilde{Q}_{zm} & \tilde{Q}_{zz} \end{bmatrix} \tag{V.243}$$

$$F_\xi = \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} \tag{V.244}$$

the block elements of $\tilde{Q}_{\xi\xi}$ and $H_{\sigma u}$ are found as solutions of the hierarchical matrix equations

$$H_{zz}D_{zz} - F_zH_{zz} = \tilde{Q}_{zz} - \check{Q}_{zz} \tag{V.245}$$

$$H_{zm}D_{\mu z} + H_{zz}D_\eta - F_zH_{zm} = \tilde{Q}_{zm} - \check{Q}_{zm} \tag{V.246}$$

$$H_{zm}D_\rho - F_zH_{z\mu} = \tilde{Q}_{z\mu} - \check{Q}_{z\mu} \tag{V.247}$$

which individually have the form needed to apply the Unstacking Theorem when solved in order, and where the structural matrices D_{zz}, $D_{\mu z}$, D_η and D_ρ are given in Appendix A.

Proof. From Appendix A we have

$$D_\xi = \begin{bmatrix} 0 & 0 & 0 \\ D_\rho & D_{\mu z} & 0 \\ 0 & D_\eta & D_{zz} \end{bmatrix} \tag{V.248}$$

which we plug into equation V.240 along with equation V.244 and get the desired result by direct calculation. We note that, although some of the μ components of $\check{Q}_{\xi\xi}$ and $H_{\xi\xi}$ were arbitrary, zero was an acceptable solution, which is what was chosen. ◃

Finally, we block partition our uncontrollable/uncontrollable normal form matrix as

$$\check{Q}_{\sigma u} = \begin{bmatrix} 0 & 0 & 0 \\ Q_{zP_1} & Q_{zP_2} & Q_{zP_3} \\ Q_{wP_1} & Q_{wP_2} & Q_{wP_3} \end{bmatrix} \tag{V.249}$$

and block partition our linearly uncontrollable quadratic state vector as

$$\sigma^{(2)} = \begin{bmatrix} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \\ \begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix} \\ w^{(2)} \end{bmatrix} \tag{V.250}$$

which we will need for the complete normal form.

9. Overall Normal Form

The overall normal form is found by assembling all six equations, (V.155), (V.158), (V.164), (V.207), (V.219), and (V.249), and plugging into equation V.144, which we restate here as

$$\begin{aligned} \dot{\chi} &= F\chi + Gv + \left(\check{f}^{(2)}(\chi) - G\check{f}_{\nu}^{(2)}(\chi)\right) \qquad \text{(V.251)} \\ &+ \left(\vec{f}^{(3)}(\chi) - \check{g}^{(1)}(\chi)\check{f}_{\nu}^{(2)}(\chi)\right) + \left(\vec{g}^{(2)}(\chi) - \check{g}^{(1)}(\chi)\check{g}_{\nu}^{(1)}(\chi)\right)v + O^{(4+)} \end{aligned}$$

Omitting the cubic terms for clarity, we can express equation V.251 in block form as

$$\begin{aligned} \begin{bmatrix} \dot{\mu} \\ \dot{z} \\ \dot{w} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ F_{\mu} & F_{z} & 0 & 0 \\ 0 & 0 & F_{w} & 0 \\ 0 & 0 & 0 & A \end{bmatrix} \begin{bmatrix} \mu \\ z \\ w \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ B \end{bmatrix} v \qquad \text{(V.252)} \\ &+ \begin{bmatrix} 0 \\ Q_{zP} \\ Q_{wP} \\ 0 \end{bmatrix} \sigma^{(2)} + \begin{bmatrix} 0 \\ Q_{zm} \\ Q_{wm} \\ 0 \end{bmatrix} \sigma y_1 + \begin{bmatrix} 0 \\ Q_{zc} \\ Q_{wc} \\ Q_{yc} \end{bmatrix} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} + O^{(3+)} \end{aligned}$$

Alternatively, we can express equation V.251 as separate equations, given by equations V.3, V.4, V.6 and V.7, which is the preferred method for the rest of this dissertation.

D. REVERSING THE TRANSFORMATIONS

Now that we have a system in quadratic normal form, we have a way to determine the control law needed to stabilize the system, or to decide if the system is unstabilizable. The control law needed will have the form of state feedback of our linear and quadratic state vectors, multiplied by appropriate gains, of the form

$$v = K_\chi^T \chi + K_{\chi^{(2)}}^T \chi^{(2)} \tag{V.253}$$

where χ and $\chi^{(2)}$ are the linear and quadratic appended state vectors after all quadratic transformations have been made, and K_χ and $K_{\chi^{(2)}}$ are vectors of the appropriate linear and quadratic state feedback gains. Picking the appropriate values of the gains is the subject of Chapters VI and VII. However, even given the proper gains, a control law of the form of equation V.253 is of little use to us, since what we need is a control law for $\breve{u}$, the control input in our original system. The purpose of this section is to show how to reverse all of the translations, transformations and feedback we have imposed on our original system to recover a control law for our original system which will implement equation V.253. We proceed in the reverse of the order in which we imposed them.

We start by reversing the quadratic order feedback we imposed on the system in equation V.43, which we repeat here.for convenience

$$\tilde{u} = v - \breve{g}_\nu^{(1)}(\chi)\, v - \breve{f}_\nu^{(2)}(\chi) \tag{V.254}$$

Plugging in equation V.253 we get

$$\tilde{u} = K_\chi^T \chi + \left(K_{\chi^{(2)}}^T \chi^{(2)} - \breve{g}_\nu^{(1)}(\chi)\, K_\chi^T \chi - \breve{f}_\nu^{(2)}(\chi)\right) - \breve{g}_\nu^{(1)}(\chi)\, K_{\chi^{(2)}}^T \chi^{(2)} \tag{V.255}$$

where we have grouped the quadratic terms together. Now we would like to reverse the quadratic coordinate transformation we imposed on the system in equation V.12, which we repeat here for convenience

$$\begin{aligned} \tilde{\chi} &= \chi + h^{(2)}(\chi) \\ &= \chi + H\chi^{(2)} \end{aligned} \tag{V.256}$$

However, we have a minor problem. Right now, we have $\tilde{\chi}$ as a function of χ, but what we want is the opposite. So, we have to invert equation V.256. If we let

$$\begin{aligned} \chi &= \Phi(\tilde{\chi}) \\ &= \Phi^{(1)}(\tilde{\chi}) + \Phi^{(2)}(\tilde{\chi}) + O^{(3+)} \end{aligned} \tag{V.257}$$

then we can plug equation V.257 into equation V.256 and solve term by term. We get

$$\Phi^{(1)}(\tilde{\chi}) = \tilde{\chi} \tag{V.258}$$

$$\Phi^{(2)}(\tilde{\chi}) = -H\chi^{(2)} \tag{V.259}$$

Plugging into equation V.255, we get

$$\tilde{u} = K_\chi^T \tilde{\chi} + \left(K_{\chi^{(2)}}^T \tilde{\chi}^{(2)} - \check{g}_\nu^{(1)}(\tilde{\chi}) K_\chi^T \tilde{\chi} - \check{f}_\nu^{(2)}(\tilde{\chi}) - K_\chi^T H \tilde{\chi}^{(2)} \right) \tag{V.260}$$

where we have neglected terms of $O^{(3+)}$. Now we can reverse the linear feedback we imposed in Chapter III, which we repeat here for convenience

$$\begin{aligned} u &= \tilde{u} - \alpha^T \mu - a^T \tilde{y} \\ &= \tilde{u} - \begin{bmatrix} \alpha^T & 0 & 0 & a^T \end{bmatrix} \tilde{\chi} \end{aligned} \tag{V.261}$$

Plugging equation V.260 into equationV.261, we get

$$\begin{aligned} u &= \left(K_\chi^T - \begin{bmatrix} \alpha^T & 0 & 0 & a^T \end{bmatrix} \right) \tilde{\chi} \\ &+ \left(K_{\chi^{(2)}}^T \tilde{\chi}^{(2)} - \left(G_\nu^T \tilde{\chi} \right) \left(K_\chi^T \tilde{\chi} \right) - F_\nu^T \tilde{\chi}^{(2)} - K_\chi^T H \tilde{\chi}^{(2)} \right) \end{aligned} \tag{V.262}$$

Now we can reverse the linear coordinate transformation we imposed in Chapter III, which we rewrite as

$$\hat{\chi} = T\tilde{\chi} \tag{V.263}$$

$$\tilde{\chi} = T^{-1}\hat{\chi} \tag{V.264}$$

where the vector $\hat{\chi}$ is defined as

$$\hat{\chi} = \begin{bmatrix} \mu \\ x \end{bmatrix} \tag{V.265}$$

This brings up the question of how to handle the quadratic state vector $\tilde{\chi}^{(2)}$, since the linear elements which are multiplied together to make up the quadratic components are being transformed. This is a difficult problem in general, but is one which is tractable (though still complicated) for a specific problem. We define a new matrix $\Upsilon\left(T^{-1}\right) \in R^{\frac{\nu(\nu+1)}{2} \times \frac{\nu(\nu+1)}{2}}$ by the relation

$$\tilde{\chi}^{(2)} \equiv \Upsilon\left(T^{-1}\right) \hat{\chi}^{(2)} \tag{V.266}$$

where the elements of Υ are determined from the elements of the matrix T^{-1} according to the following relation,

$$\tilde{\chi}_i \tilde{\chi}_j = \left(T_{i1}^{-1} \hat{\chi}_1 + \ldots + T_{i\nu}^{-1} \hat{\chi}_\nu\right)\left(T_{j1}^{-1} \hat{\chi}_1 + \ldots + T_{j\nu}^{-1} \hat{\chi}_\nu\right) \tag{V.267}$$

We illustrate with an example.

Example. Given the linear coordinate transformation

$$\hat{x} = T\tilde{x} \tag{V.268}$$

such that

$$T^{-1} = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \tag{V.269}$$

we would like to find $\Upsilon\left(T^{-1}\right)$ such that

$$\tilde{x}^{(2)} = \Upsilon\left(T^{-1}\right) \hat{x}^{(2)} \tag{V.270}$$

So, we calculate each row of $\tilde{x}^{(2)}$ in turn.

$$\begin{aligned} \tilde{x}_1^2 &= \left(t_1\hat{x}_1 + t_2\hat{x}_2\right)^2 \\ &= t_1^2\hat{x}_1^2 + 2t_1t_2\hat{x}_1\hat{x}_2 + t_2^2\hat{x}_2^2 \\ &= \begin{bmatrix} t_1^2 & 2t_1t_2 & t_2^2 \end{bmatrix} \hat{x}^{(2)} \end{aligned} \tag{V.271}$$

$$\begin{aligned} \tilde{x}_1\tilde{x}_2 &= \left(t_1\hat{x}_1 + t_2\hat{x}_2\right)\left(t_3\hat{x}_1 + t_4\hat{x}_2\right) \\ &= t_1t_3\hat{x}_1^2 + \left(t_1t_4 + t_2t_3\right)\hat{x}_1\hat{x}_2 + t_2t_4\hat{x}_2^2 \\ &= \begin{bmatrix} t_1t_3 & \left(t_1t_4 + t_2t_3\right) & t_2t_4 \end{bmatrix} \hat{x}^{(2)} \end{aligned} \tag{V.272}$$

$$
\begin{aligned}
\tilde{x}_2^2 &= \left(t_3\hat{x}_1 + t_4\hat{x}_2\right)^2 && \text{(V.273)} \\
&= t_3^2\hat{x}_1^2 + 2t_3t_4\hat{x}_1\hat{x}_2 + t_4^2\hat{x}_2^2 \\
&= \left[\begin{array}{ccc} t_3^2 & 2t_3t_4 & t_4^2 \end{array}\right]\hat{x}^{(2)}
\end{aligned}
$$

Now, stacking up the rows, we get

$$
\tilde{x}^{(2)} = \left[\begin{array}{c} \tilde{x}_1^2 \\ \tilde{x}_1\tilde{x}_2 \\ \tilde{x}_2^2 \end{array}\right] = \left[\begin{array}{ccc} t_1^2 & 2t_1t_2 & t_2^2 \\ t_1t_3 & (t_1t_4 + t_2t_3) & t_2t_4 \\ t_3^2 & 2t_3t_4 & t_4^2 \end{array}\right]\hat{x}^{(2)} \tag{V.274}
$$

So, the answer for $\Upsilon\left(T^{-1}\right)$ is

$$
\Upsilon\left(T^{-1}\right) = \left[\begin{array}{ccc} t_1^2 & 2t_1t_2 & t_2^2 \\ t_1t_3 & (t_1t_4 + t_2t_3) & t_2t_4 \\ t_3^2 & 2t_3t_4 & t_4^2 \end{array}\right] \tag{V.275}
$$

This example is illustrative of the process required to calculate the matrix $\Upsilon\left(T^{-1}\right)$ for a specific problem. ◁

Now, we can transform equation V.262 by applying equations V.264 and V.266. We get

$$
\begin{aligned}
u &= \left(K_\chi^T - \left[\begin{array}{cccc} \alpha^T & 0 & 0 & a^T \end{array}\right]\right)T^{-1}\hat{\chi} && \text{(V.276)} \\
&- \left(G_\nu^T T^{-1}\hat{\chi}\right)\left(K_\chi^T T^{-1}\hat{\chi}\right) + \left(K_{\chi^{(2)}}^T - F_\nu^T - K_\chi^T H\right)\Upsilon\left(T^{-1}\right)\hat{\chi}^{(2)}
\end{aligned}
$$

Equation V.276 gives the control law for our original system after the origin was shifted to the equilibrium point at the point of bifurcation. Often this will be sufficient for many applications. If the control law is required in the original untranslated system, then the reversed coordinate translations

$$
\hat{\chi} = \left[\begin{array}{c} \mu \\ x \end{array}\right] = \left[\begin{array}{c} \check{\mu} - \check{\mu}^* \\ \check{x} - \check{x}^* \end{array}\right] \tag{V.277}
$$

should be plugged into equation V.276 by individual components and multiplied out.

VI. CONTROL OF CENTER MANIFOLD

A. ROADMAP: THE BIG PICTURE

1. Results of Previous Chapters

Chapters II, III and V showed that any affine system

$$\dot{\breve{x}} = \breve{f}(\breve{x}, \breve{\mu}) + \breve{g}(\breve{x}, \breve{\mu})\breve{u} \tag{VI.1}$$

can be put into quadratic normal form

$$\dot{\mu} = 0 \tag{VI.2}$$

$$\begin{aligned}
\dot{z} &= F_\mu \mu + F_z z \\
&+ Q_{z_{P_1}} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + Q_{z_{m_1}} \begin{bmatrix} \mu y_1 \\ z y_1 \end{bmatrix} + Q_{z_c} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \\
&+ Q_{z_{P_2}} \begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix} + Q_{z_{P_3}} \begin{bmatrix} w^{(2)} \end{bmatrix} + Q_{z_{m_2}} \begin{bmatrix} w y_1 \end{bmatrix} \\
&+ f_z^{(3)}(\mu, z, w, y) + g_z^{(2)}(\mu, z, w, y)\, v + O^{(4+)}
\end{aligned} \tag{VI.3}$$

$$\begin{aligned}
\dot{w} &= F_w w \\
&+ Q_{w_{P_1}} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + Q_{w_{m_1}} \begin{bmatrix} \mu y_1 \\ z y_1 \end{bmatrix} + Q_{w_c} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \\
&+ Q_{w_{P_2}} \begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix} + Q_{w_{P_3}} \begin{bmatrix} w^{(2)} \end{bmatrix} + Q_{w_{m_2}} \begin{bmatrix} w y_1 \end{bmatrix} + O^{(3+)}
\end{aligned} \tag{VI.4}$$

$$\dot{y} = Ay + Bv + Q_{y_c} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} + O^{(3+)} \tag{VI.5}$$

where $\mu \in R^r$ is the vector of parameters, $z \in R^q$ is the vector of linearly uncontrollable states having zero real-part eigenvalues, $w \in R^m$ is the vector of linearly uncontrollable states having non-zero real-part eigenvalues, $y \in R^p$ is the vector of linearly controllable states, $v \in R^1$ is a single control input, and A and B are the appropriately scaled Brunovsky form matrices. We can pick the control $v = v(\mu, z, w, y)$ as state feedback. It's expansion through quadratic order is

$$\begin{aligned} v &= K_\mu^T \mu + K_z^T z + K_w^T w + K_y^T y \\ &+ K_{\mu(2)}^T \mu^{(2)} + K_{\mu z(2)}^T \mu z^{(2)} + K_{z(2)}^T z^{(2)} + K_{\mu w(2)}^T \mu w^{(2)} + K_{zw(2)}^T z w^{(2)} + K_{w(2)}^T w^{(2)} \\ &+ K_{\mu y(2)}^T \mu y^{(2)} + K_{zy(2)}^T z y^{(2)} + K_{wy(2)}^T w y^{(2)} + K_{y(2)}^T y^{(2)} \end{aligned} \tag{VI.6}$$

where the vector of gains K have yet to be chosen.

2. Purpose of this Chapter

In this chapter we will consider how to control the shape of the center manifold of our linearly uncontrollable dynamic system by properly choosing our vector of feedback gains K. That is, after all the exponentially stable modes die out, the remaining dynamics of our system are restricted to a surface of reduced dimension — the center manifold — and we will show how to pick our feedback gains to control the shape of this surface. Controlling the shape of the center manifold surface is what will allow us to control bifurcations occurring in the linearly uncontrollable states μ and z in the next chapter. (Note that we assume that all of the linearly uncontrollable states w are exponentially stable — otherwise our system diverges from the origin — and that we will stabilize the linearly controllable states y with appropriate state feedback so that they are also exponentially stable.) The center manifold is defined

by the relations

$$w_{cm} = \Omega(\mu, z) \tag{VI.7}$$

$$y_{cm} = \Pi(\mu, z) \tag{VI.8}$$

where we have used the notation w_{cm} and y_{cm} to indicate the values of w and y on the center manifold surface. The uncontrollable center manifold function $\Omega(\mu, z)$ and the controllable center manifold function $\Pi(\mu, z)$ have Taylor series expansions of the form

$$\Omega(\mu, z) = \Omega_L \begin{bmatrix} \mu \\ z \end{bmatrix} + \Omega_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \tag{VI.9}$$

$$\Pi(\mu, z) = \Pi_L \begin{bmatrix} \mu \\ z \end{bmatrix} + \Pi_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \tag{VI.10}$$

We will give expressions for the linear and quadratic coefficient matrices Ω_L and Ω_Q, and we will show that the linear and quadratic coefficient matrices Π_L and Π_Q are functions of their first row only, given by the formulas

$$\Pi_{Li} = \Pi_{L1} \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^{i-1} \tag{VI.11}$$

and

$$\Pi_{Qi} = \Pi_{Q1} D_\xi^{i-1} - \sum_{j=1}^{i-1} \Gamma_{zj}(\Pi_L) D_\xi^{i-j-1} \tag{VI.12}$$

where the subscript i indicates the row in question, where D_ξ is a structural matrix defined in Appendix A and where $\Gamma_z(\Pi_L)$ is a matrix which will be defined later in this chapter. We will also show that the vector of gains K can be chosen to produce a desired center manifold function $\Pi(\mu, z)$ with the formulas

$$\begin{bmatrix} K_\mu^T & K_z^T \end{bmatrix} = \Pi_{L_1} \left(\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^p - \sum_{i=1}^{p} K_{y_i} \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^{i-1} \right) \tag{VI.13}$$

and

$$\left[\begin{array}{ccc} K^T_{\mu(2)} & K^T_{\mu z(2)} & K^T_{z(2)} \end{array}\right] = \Pi_{Q_1} D_K + \Gamma^T_K \tag{VI.14}$$

where K_μ, K_z and K_y are vectors of linear state feedback gains, $K^T_{\mu(2)}$, $K^T_{\mu z(2)}$ and $K^T_{z(2)}$ are vectors of quadratic state feedback gains, and the matrices D_K and Γ^T_K will be defined later. These relations will be needed in the next chapter.

B. DETERMINING THE CENTER MANIFOLD SURFACE

What is a center manifold? In Chapter I we stated that a center manifold is a surface contained in a state space which has two properties:

- The surface is invariant, that is, trajectories which begin on any point on the surface stay on the surface.
- The surface is tangent to the "center subspace" of the linearization around the origin, where the center subspace is defined as the subspace spanned by the generalized eigenvectors having eigenvalues with zero real parts.

Now at this point, those two properties may not be very illuminating. However, a simple way to think of the center manifold is to imagine a surface passing through the origin. In the case of a one dimensional center manifold, the surface is a curve. In the case of a two dimensional center manifold, the surface is a sheet. (It is left as an exercise to the reader to visualize a three dimensional center manifold inside a higher dimensional state space. In any event, we will only deal with one and two dimensional center manifolds in this dissertation.) If we pick a point on the center manifold surface, then the trajectory the point follows has to stay on the surface. The trajectory is also limited by the fact that the eigenvalues of the linearization at the origin have zero real parts, which imposes additional constraints on the trajectories possible. We now illustrate with some examples.

Example. [One Dimensional Center Manifold] A one dimensional center manifold is a curve passing through the origin. Because it is one dimensional, the linearization at the origin has only one eigenvalue, which must be zero. Therefore,

trajectories on a one dimensional center manifold are capable of only algebraic decay toward the origin, or algebraic growth away from the origin, since the rate of growth or decay is determined by the higher order non-linear terms. Points which are not on the center manifold, but are displaced slightly from it, exponentially decay toward points on the center manifold. The points on the center manifold are decaying or growing algebraically, which is much slower than exponential decay, so all points slightly displaced from a one dimensional center manifold essentially collapse rapidly to the center manifold, and then slowly grow or decay on it. ◃

Example. [Two Dimensional Center Manifold, Part I] A two dimensional center manifold is a sheet passing through the origin. Because the center manifold is two dimensional, there are two ways to have eigenvalues with zero real parts. The linearization can have two eigenvalues which are complex conjugates, with zero real parts, as is illustrated in Figure 7. Or, the linearization can have one eigenvalue,

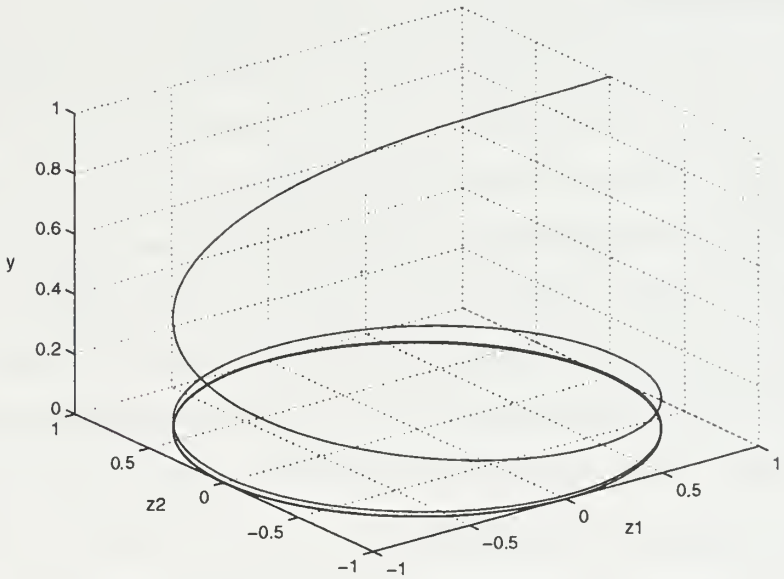

Figure 7. Collapse to 2-D Center Manifold

which is zero, with either one actual eigenvector and one generalized eigenvector, or with two generalized eigenvectors. In the case of two complex conjugate eigenvalues, points on a two dimensional center manifold follow essentially circular trajectories

around the origin, with the frequency of rotation being determined by the linearization, but with the radius capable of only algebraic decay toward the origin or algebraic growth away from the origin, since the rate of growth or decay is determined by the higher order non-linear terms. That is, trajectories on this type of center manifold are slow spirals toward or away from the origin. Points which are not on the center manifold, but are displaced slightly from it, exponentially decay toward the center manifold surface. However, during the decay, their trajectories have to match the rotation rate of points on the center manifold, so all points slightly displaced from a two dimensional center manifold with complex conjugate eigenvalues follow essentially helical trajectories which collapse rapidly to essentially circular trajectories on the center manifold, which then slowly spiral toward or away from the origin on the center manifold surface. ◁

Example. [Two Dimensional Center Manifold, Part II] For the case of a two dimensional center manifold with a single zero eigenvalue, points on the center manifold surface follow trajectories in which each of two components algebraically decays or algebraically grows independently. The rate of growth or decay is determined by the higher order non-linear terms. Points which are not on the center manifold surface, but are displaced slightly from it, exponentially decay toward points on the center manifold. The points on the center manifold are decaying and/or growing algebraically, which is much slower than exponential decay, so all points slightly displaced from a two dimensional center manifold with a single zero eigenvalue essentially collapse rapidly to the center manifold, then follow trajectories on the surface which slowly decay to the origin or diverge from it. ◁

Now, for our specific case, near the origin, both the linearly uncontrollable states w and the linearly controllable states y are assumed to be exponentially stable, while the magnitude of the linearly uncontrollable states z are capable of at best only algebraic decay or growth, since the real parts of the eigenvalues of their linearization are zero. (Note that the specific values of the individual components of the states

z on the center manifold may change quite rapidly, as for example on a limit cycle trajectory, but that their magnitude is restricted to algebraic growth or decay.) So, we have a situation where we can think of the dynamics of the magnitude of z as happening on a much slower time scale than the dynamics of the magnitudes of w and y. That is, near the origin, if we think of z as being on a fixed trajectory or having a fixed value, and given that μ has a fixed value, then w and y both collapse to "fixed" trajectories or "fixed" values which depend on the values or magnitudes of μ and z. This functional dependence of w and y (after their exponential decay) on μ and z defines the shape of the center manifold surface, which is given by

$$w_{cm} = \Omega(\mu, z) \quad \text{(VI.15)}$$

$$y_{cm} = \Pi(\mu, z) \quad \text{(VI.16)}$$

where we have used the notation w_{cm} and y_{cm} to indicate the values of w and y on the center manifold surface, and where $\Omega(\mu, z)$ and $\Pi(\mu, z)$ are vector valued functions which we are trying to determine. We can expand each of them as a Taylor series around the origin to yield

$$w_{cm} = \Omega_L \begin{bmatrix} \mu \\ z \end{bmatrix} + \Omega_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \quad \text{(VI.17)}$$

$$y_{cm} = \Pi_L \begin{bmatrix} \mu \\ z \end{bmatrix} + \Pi_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \quad \text{(VI.18)}$$

Now, following Carr [Ref. 17], we can determine the as-yet-unknown constant coefficient matrices Ω_L, Ω_Q, Π_L, and Π_Q by calculating the trajectories of the system two different ways, and equating them. In the first way, we calculate the dynamics of the system based on the gradient of the center manifold, which are determined by the z dynamics given by equation VI.3. In the second way, we calculate the dynamics of the system anywhere, and then restrict them to the center manifold, which are

determined by the w and y dynamics given by equations VI.4 and VI.5. At those points where the dynamics from the two ways are equal, there is our center manifold — and since we have control of our y dynamics through the gain vector K, we have the possibility of affecting the controllable part of the center manifold. Now we look at the two ways of calculating the trajectories of the system.

1. Dynamics Based on the Center Manifold Gradient

The first way to calculate trajectories of our system is to calculate the dynamics of the system based on the gradient of the center manifold surface. On the center manifold, which is where all the dynamics end up after the exponentially stable states have decayed away, the dynamics are determined by

$$\dot{\Omega} = \frac{\partial \Omega(\mu, z)}{\partial \mu}\dot{\mu} + \frac{\partial \Omega(\mu, z)}{\partial z}\dot{z} \qquad \text{(VI.19)}$$

$$\dot{\Pi} = \frac{\partial \Pi(\mu, z)}{\partial \mu}\dot{\mu} + \frac{\partial \Pi(\mu, z)}{\partial z}\dot{z} \qquad \text{(VI.20)}$$

which, since $\dot{\mu} = 0$, and using our Taylor series expansions, can be expressed as

$$\begin{aligned} \dot{\Omega} &= \frac{\partial \Omega(\mu, z)}{\partial z}\dot{z} = \Omega_L \frac{\partial}{\partial z}\begin{bmatrix} \mu \\ z \end{bmatrix}\dot{z} + \Omega_Q \frac{\partial}{\partial z}\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix}\dot{z} + O^{(3+)} \qquad \text{(VI.21)} \\ &= \Omega_L \begin{bmatrix} 0 \\ I \end{bmatrix}\dot{z} + \Omega_Q \frac{\partial}{\partial z}\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix}\dot{z} + O^{(3+)} \end{aligned}$$

and

$$\begin{aligned} \dot{\Pi} &= \frac{\partial \Pi(\mu, z)}{\partial z}\dot{z} = \Pi_L \frac{\partial}{\partial z}\begin{bmatrix} \mu \\ z \end{bmatrix}\dot{z} + \Pi_Q \frac{\partial}{\partial z}\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix}\dot{z} + O^{(3+)} \qquad \text{(VI.22)} \\ &= \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix}\dot{z} + \Pi_Q \frac{\partial}{\partial z}\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix}\dot{z} + O^{(3+)} \end{aligned}$$

Plugging in the z dynamics from equation VI.3 gives us

$$
\begin{aligned}
\dot{\Omega} &= \Omega_L \begin{bmatrix} 0 \\ I \end{bmatrix} (F_\mu \mu + F_z z) \qquad \text{(VI.23)} \\
&+ \Omega_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{zP_1} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + Q_{zm_1} \begin{bmatrix} \mu y_1 \\ z y_1 \end{bmatrix} + Q_{zc} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \right) \\
&+ \Omega_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{zP_2} \begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix} + Q_{zP_3} \begin{bmatrix} w^{(2)} \end{bmatrix} + Q_{zm_2} \begin{bmatrix} w y_1 \end{bmatrix} \right) \\
&+ \Omega_Q \frac{\partial}{\partial z} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} (F_\mu \mu + F_z z) + O^{(3+)}
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\Pi} &= \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix} (F_\mu \mu + F_z z) \qquad \text{(VI.24)} \\
&+ \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{zP_1} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + Q_{zm} \begin{bmatrix} \mu y_1 \\ z y_1 \end{bmatrix} + Q_{zc} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \right) \\
&+ \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{zP_2} \begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix} + Q_{zP_3} \begin{bmatrix} w^{(2)} \end{bmatrix} + Q_{zm_2} \begin{bmatrix} w y_1 \end{bmatrix} \right) \\
&+ \Pi_Q \frac{\partial}{\partial z} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} (F_\mu \mu + F_z z) + O^{(3+)}
\end{aligned}
$$

where we note that we still have to plug in for $w = \Omega(\mu, z)$ and $y = \Pi(\mu, z)$ in the quadratic terms in each equation. However, first notice that the terms involving the

partial derivatives can be expressed in terms of the structural matrix D_ξ (developed more fully in Appendix A) which we first encountered in Chapter V in deriving the quadratic normal form. That is

$$\frac{\partial}{\partial z}\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix}(F_\mu \mu + F_z z) = D_\xi \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \tag{VI.25}$$

Next, we define additional structural matrices to support our derivation of dynamics on the center manifold. We define

$$\begin{bmatrix} \mu y_1 \\ z y_1 \end{bmatrix}_{cm} \equiv M_1(\Pi_L)\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_1(\Pi_L, \Pi_Q)\begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{VI.26}$$

$$\begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix}_{cm} \equiv M_2(\Pi_L)\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_2(\Pi_L, \Pi_Q)\begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{VI.27}$$

$$\begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix}_{cm} \equiv M_3(\Omega_L)\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_3(\Omega_L, \Omega_Q)\begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{VI.28}$$

$$\begin{bmatrix} w^{(2)} \end{bmatrix}_{cm} \equiv M_4(\Omega_L)\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_4(\Omega_L, \Omega_Q)\begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{VI.29}$$

$$\begin{bmatrix} w y_1 \end{bmatrix}_{cm} \equiv M_5(\Omega_L, \Pi_L)\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_5(\Omega_L, \Omega_Q, \Pi_L, \Pi_Q)\begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{VI.30}$$

where the notation $[V(\mu, z, w, y)]_{cm}$ indicates the vector obtained by substituting $w = \Omega(\mu, z)$ and $y = \Pi(\mu, z)$ into V. The matrices M_1, M_2, M_3, M_4, M_5, N_1, N_2,

N_3, N_4 and N_5 are functions of their arguments, and are developed more fully in Appendix C. So, now we can plug into our equations for the final result

$$\begin{aligned}
\dot{\Omega} &= \Omega_L \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} \begin{bmatrix} \mu \\ z \end{bmatrix} \\
&+ \Omega_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{zP_1} + Q_{zm_1} M_1 (\Pi_L) + Q_{zc} M_2 (\Pi_L) \right) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \\
&+ \Omega_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{zP_2} M_3 (\Omega_L) + Q_{zP_3} M_4 (\Omega_L) + Q_{zm_2} M_5 (\Omega_L, \Pi_L) \right) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \\
&+ \Omega_Q D_\xi \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)}
\end{aligned} \tag{VI.31}$$

and

$$\begin{aligned}
\dot{\Pi} &= \Pi_L \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} \begin{bmatrix} \mu \\ z \end{bmatrix} \\
&+ \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{zP_1} + Q_{zm_1} M_1 (\Pi_L) + Q_{zc} M_2 (\Pi_L) \right) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \\
&+ \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{zP_2} M_3 (\Omega_L) + Q_{zP_3} M_4 (\Omega_L) + Q_{zm_2} M_5 (\Omega_L, \Pi_L) \right) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \\
&+ \Pi_Q D_\xi \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)}
\end{aligned} \tag{VI.32}$$

Equations VI.31 and VI.32 are the first set of equations we will use to determine $\Omega(\mu, z)$ and $\Pi(\mu, z)$.

2. Dynamics Restricted to the Center Manifold

The second way to calculate trajectories of our system is to calculate the dynamics of our system anywhere, and then restrict the location to the center manifold surface. That is, lets look at the dynamics of w and y, and ask what they would be if they happened to be on the center manifold. Our w and y dynamics are given by equations VI.4 and VI.5, which we restate here for convenience

$$\begin{aligned} \dot{w} &= F_w w \\ &+ Q_{w_{P_1}} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + Q_{w_{m_1}} \begin{bmatrix} \mu y_1 \\ z y_1 \end{bmatrix} + Q_{w_c} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \\ &+ Q_{w_{P_2}} \begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix} + Q_{w_{P_3}} \begin{bmatrix} w^{(2)} \end{bmatrix} + Q_{w_{m_2}} \begin{bmatrix} w y_1 \end{bmatrix} + O^{(3+)} \end{aligned} \tag{VI.33}$$

and

$$\dot{y} = Ay + Bv + Q_{y_c} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} + O^{(3+)} \tag{VI.34}$$

Now, if we plug in for the case when w and y happen to be on the center manifold, that is when $w = w_{cm}$ and $y = y_{cm}$, we get

$$\begin{aligned} \dot{w}_{cm} &= F_w \Omega_L \begin{bmatrix} \mu \\ z \end{bmatrix} + F_w \Omega_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \\ &+ \left(Q_{w_{P_1}} + Q_{w_{m_1}} M_1 (\Pi_L) + Q_{w_c} M_2 (\Pi_L) \right) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \end{aligned} \tag{VI.35}$$

$$+ \quad \left(Q_{w_{P_2}} M_3\left(\Omega_L\right) + Q_{w_{P_3}} M_4\left(\Omega_L\right) + Q_{w_{m_2}} M_5\left(\Omega_L, \Pi_L\right)\right) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix}$$

$$+ \quad O^{(3+)}$$

$$\dot{y}_{cm} = A\Pi_L \begin{bmatrix} \mu \\ z \end{bmatrix} + A\Pi_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + Bv + Q_{y_c} M_2\left(\Pi_L\right) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \quad \text{(VI.36)}$$

where we have plugged in w_{cm} and y_{cm} from equations VI.17 and VI.18, and have used equations VI.26 through VI.30. So, equations VI.35 and VI.36 are the second set of equations we will use to determine $\Omega(\mu, z)$ and $\Pi(\mu, z)$.

3. Center Manifold Theorem

Now, we determine $\Omega(\mu, z)$ and $\Pi(\mu, z)$ by setting equations VI.31 and VI.32 equal to equations VI.35 and VI.36 respectively. We get

$$\dot{w}_{cm} = \dot{\Omega} \quad \text{(VI.37)}$$

and

$$\dot{y}_{cm} = \dot{\Pi} \quad \text{(VI.38)}$$

This allows us to match the dynamics term by term, and so determine the linear and quadratic center manifold coefficient matrices Ω_L, Ω_Q, Π_L, and Π_Q. We present the results in the Center Manifold Theorem, after stating two useful lemmas.

Lemma B.1 (Sylvester Equation) *A matrix equation of the form*

$$\Phi C + D\Phi = E \quad \text{(VI.39)}$$

with C and D square matrices of possibly different dimensions, is called a Sylvester equation, and has a unique solution matrix Φ for each matrix E if and only if the matrices C and D have no eigenvalues in common.

Proof. This lemma is proven in Appendix A of Knobloch [Ref. 19]. ◃

Lemma B.2 (Eigenvalues of D_ξ) *The structural matrix D_ξ, defined by the relation*

$$D_\xi \xi^{(2)} \equiv \frac{\partial \xi^{(2)}}{\partial \xi} F_\xi \xi \tag{VI.40}$$

and given in Appendix A by the block formula

$$D_\xi = \begin{bmatrix} 0 & 0 & 0 \\ D_\rho & D_{\mu z} & 0 \\ 0 & D_\eta & D_{zz} \end{bmatrix} \tag{VI.41}$$

has only eigenvalues with zero real parts.

Proof. The eigenvalues of D_ξ are determined by the eigenvalues of the matrices 0, $D_{\mu z}$ and D_{zz}, since D_ξ is block diagonal. The eigenvalues of the 0 matrix are zero. The matrix $D_{\mu z}$ is given in Appendix A by the general formula

$$D_{\mu z} = \begin{bmatrix} F_{z_{11}} I & \cdots & F_{z_{1q}} I \\ \vdots & & \vdots \\ F_{z_{q1}} I & \cdots & F_{z_{qq}} I \end{bmatrix} \tag{VI.42}$$

Since the matrix F_z is block diagonal, then the matrix $D_{\mu z}$ is block diagonal if the matrices $F_{z_{ij}} I$ are taken as elements of a block. There are three types of blocks. Distinct real eigenvalues of F_z produce blocks on the main diagonal of $D_{\mu z}$ of the form $F_{z_{jj}} I$, with $F_{z_{jj}} = 0$, since that is the only possible real eigenvalue with a zero real-part. The eigenvalues of these blocks are zero. Non-distinct real eigenvalues of F_z produce Jordan blocks on the main diagonal of $D_{\mu z}$ of the form

$$J_i = \begin{bmatrix} F_{z_{ii}} I & I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I \\ 0 & \cdots & \cdots & 0 & F_{z_{ii}} I \end{bmatrix} \tag{VI.43}$$

with $F_{z_{ii}} = 0$, since that is the only possible real eigenvalue with a zero real-part. The eigenvalues of these blocks are zero. Complex conjugate pairs of eigenvalues of

F_z produce blocks on the main diagonal of $D_{\mu z}$ of the form

$$J_k = \begin{bmatrix} 0 & -F_{z_{kk}} I \\ F_{z_{kk}} I & 0 \end{bmatrix} \tag{VI.44}$$

with $F_{z_{kk}} \neq 0$. The eigenvalues of these blocks are complex conjugate pairs, $0 \pm i F_{z_{kk}}$, which have zero real parts. Therefore, all eigenvalues of the matrix $D_{\mu z}$ have zero real parts. Finally, D_{zz} is seen to have zero real-part eigenvalues by examining the cases given in Appendix A, and performing a block analysis as shown above. ◃

Theorem B.3 (Center Manifold Theorem) *For a dynamic system in the quadratic normal form given by equations VI.2, VI.3, VI.4, and VI.5, the linear and quadratic center manifold coefficient matrices Ω_L, Ω_Q, Π_L, and Π_Q of the exponentially stable states w and linearly controllable states y are defined in equations VI.17 and VI.18. They can be found by solving the following matrix equations in order for Ω_L, Π_L, Ω_Q and Π_Q:*

$$\Omega_L = 0 \tag{VI.45}$$

$$\Pi_L \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} - \left(A + BK_y^T\right)\Pi_L = B\begin{bmatrix} K_\mu^T & K_z^T \end{bmatrix} \tag{VI.46}$$

$$\Omega_Q D_\xi - F_w \Omega_Q = \Gamma_w\left(\Pi_L\right) \tag{VI.47}$$

$$\Pi_Q D_\xi - \left(A + BK_y^T\right)\Pi_Q = B\tilde{K}_{(2)}^T + \Gamma_z\left(\Pi_L\right) \tag{VI.48}$$

where K_μ, K_z and K_y are vectors of linear state feedback gains and D_ξ is a structural matrix defined in Appendix A. The vector $\tilde{K}_{(2)}$ is a specialized vector of quadratic state feedback gains defined by

$$\begin{aligned} \tilde{K}_{(2)}^T &= \begin{bmatrix} K_{\mu(2)}^T & K_{\mu z(2)}^T & K_{z(2)}^T \end{bmatrix} \\ &+ \left(K_w^T \Omega_Q + \begin{bmatrix} K_{\mu y(2)}^T & K_{zy(2)}^T \end{bmatrix} M_6\left(\Pi_L\right) + K_{y(2)}^T M_8\left(\Pi_L\right)\right) \end{aligned} \tag{VI.49}$$

where $K_{\mu(2)}$, $K_{\mu z(2)}$, $K_{z(2)}$, $K_{\mu y(2)}$, $K_{zy(2)}$ and $K_{y(2)}$ are vectors of quadratic state feedback gains, K_w is a vector of linear state feedback gains, and the matrices $M_6\left(\Pi_L\right)$ and $M_8\left(\Pi_L\right)$ are structural matrices defined in Appendix C. The matrices $\Gamma_w\left(\Pi_L\right)$, and $\Gamma_z\left(\Pi_L\right)$ are defined by the relations

$$\Gamma_w\left(\Pi_L\right) = Q_{w_{P_1}} + Q_{w_{m_1}} M_1\left(\Pi_L\right) + Q_{w_c} M_2\left(\Pi_L\right) \tag{VI.50}$$

and

$$\begin{aligned} \Gamma_z\left(\Pi_L\right) &= Q_{y_c} M_2\left(\Pi_L\right) \\ &- \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix}\left(Q_{z_{P_1}} + Q_{z_{m_1}} M_1\left(\Pi_L\right) + Q_{z_c} M_2\left(\Pi_L\right)\right) \end{aligned} \tag{VI.51}$$

where the matrices $M_1\left(\Pi_L\right)$ and $M_2\left(\Pi_L\right)$ are defined in Appendix C.

Proof. For the linear part of the w dynamics in equation VI.37 we equate the linear terms from equation VI.31 with the linear terms from equation VI.35 and get

$$F_w \Omega_L \begin{bmatrix} \mu \\ z \end{bmatrix} = \Omega_L \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} \begin{bmatrix} \mu \\ z \end{bmatrix} \tag{VI.52}$$

This can only be true when

$$F_w \Omega_L - \Omega_L \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} = 0 \tag{VI.53}$$

Since equation VI.53 is a Sylvester equation, by the Sylvester Equation lemma, it has a unique solution whenever there are no eigenvalues in common between F_w and $\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}$. Since by definition the eigenvalues of F_w are assumed to have non-zero real parts and the eigenvalues of F_z are assumed to have zero real parts, the two matrices have no eigenvalues in common. This implies that

$$\Omega_L = 0 \tag{VI.54}$$

is the unique solution to equation VI.53, which proves the first part of the theorem.

Now look at the linear part of the y dynamics in equation VI.38. Equating the linear terms from equation VI.32 with the linear terms from equation VI.36 gives

$$A\Pi_L \begin{bmatrix} \mu \\ z \end{bmatrix} + Bv^{(1)} = \Pi_L \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} \begin{bmatrix} \mu \\ z \end{bmatrix} \tag{VI.55}$$

To calculate this, we need to know the linear part of our control input, v. We use state feedback and set

$$v^{(1)} = K_\mu^T \mu + K_z^T z + K_w^T (w)^{(1)} + K_y^T (y)^{(1)} \tag{VI.56}$$

where we have used the notation

$$(w)^{(1)} = \Omega_L \begin{bmatrix} \mu \\ z \end{bmatrix} \tag{VI.57}$$

$$(y)^{(1)} = \Pi_L \begin{bmatrix} \mu \\ z \end{bmatrix} \tag{VI.58}$$

Plugging in, rearranging, and recognizing that $\Omega_L = 0$, gives our equation for the linear part of the w dynamics. We have

$$\Pi_L \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} - \left(A + BK_y^T\right)\Pi_L = B\begin{bmatrix} K_\mu^T & K_z^T \end{bmatrix} \tag{VI.59}$$

Notice that because $\Omega_L = 0$, our linear gain K_w^T did not enter into this equation. Equation VI.59 is also a Sylvester equation, and as long as the gains K_y are picked so that the closed loop matrix $\left(A + BK_y^T\right)$ is strictly stable, then there is a unique solution for Π_L by the Sylvester Equation lemma. This proves the second part of the theorem.

Now look at the quadratic w dynamics in equation VI.37. Equating the quadratic terms from equation VI.31 with the quadratic terms from equation VI.35 and plugging in $\Omega_L = 0$, we get

$$\begin{aligned} & F_w\Omega_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \\ + & \left(Q_{w_{P_1}} + Q_{w_{m_1}} M_1\left(\Pi_L\right) + Q_{w_c} M_2\left(\Pi_L\right)\right) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \\ + & \left(Q_{w_{P_2}} M_3\left(\Omega_L\right) + Q_{w_{P_3}} M_4\left(\Omega_L\right) + Q_{w_{m_2}} M_5\left(\Omega_L, \Pi_L\right)\right) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \\ = & \Omega_Q D_\xi \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \end{aligned} \tag{VI.60}$$

which is only true when

$$\begin{aligned} \Omega_Q D_\xi - F_w\Omega_Q &= Q_{w_{P_1}} + Q_{w_{m_1}} M_1\left(\Pi_L\right) + Q_{w_c} M_2\left(\Pi_L\right) \\ &+ Q_{w_{P_2}} M_3\left(\Omega_L\right) + Q_{w_{P_3}} M_4\left(\Omega_L\right) + Q_{w_{m_2}} M_5\left(\Omega_L, \Pi_L\right) \end{aligned} \tag{VI.61}$$

Now, from Appendix C, if $\Omega_L = 0$, then $M_3(\Omega_L) = 0$, $M_4(\Omega_L, \Pi_L) = 0$ and $M_5(\Omega_L, \Pi_L) = 0$. So, the quadratic term of the w dynamics simplifies to

$$\Omega_Q D_\xi - F_w \Omega_Q = Q_{w_{P_1}} + Q_{w_{m_1}} M_1(\Pi_L) + Q_{w_c} M_2(\Pi_L) \tag{VI.62}$$

and since the theorem defined

$$\Gamma_w(\Pi_L) = Q_{w_{P_1}} + Q_{w_{m_1}} M_1(\Pi_L) + Q_{w_c} M_2(\Pi_L) \tag{VI.63}$$

we have

$$\Omega_Q D_\xi - F_w \Omega_Q = \Gamma_w(\Pi_L) \tag{VI.64}$$

Since the matrix D_ξ has only eigenvalues with zero real parts by the previous lemma, and since the matrix F_w has only eigenvalues with non-zero real parts, Ω_Q has a unique solution by the Sylvester Equation lemma. This proves the third part of the theorem.

Now look at the quadratic y dynamics in equation VI.38. Equating the quadratic terms from equation VI.32 with the quadratic terms from equation VI.36, we get

$$A\Pi_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + Bv^{(2)} + Q_{y_c} M_2(\Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \tag{VI.65}$$

$$\begin{aligned}
&= \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{z_{P_1}} + Q_{z_{m_1}} M_1(\Pi_L) + Q_{z_c} M_2(\Pi_L) \right) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \\
&+ \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{z_{P_2}} M_3(\Omega_L) + Q_{z_{P_3}} M_4(\Omega_L) + Q_{z_{m_2}} M_5(\Omega_L, \Pi_L) \right) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \\
&+ \Pi_Q D_\xi \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix}
\end{aligned}$$

Looking at the quadratic control input, $v^{(2)}$, we have

$$v^{(2)} = K_w^T (w)^{(2)} + K_y^T (y)^{(2)} + \begin{bmatrix} K_{\mu^{(2)}}^T & K_{\mu z^{(2)}}^T & K_{z^{(2)}}^T \end{bmatrix} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \tag{VI.66}$$

$$+ \begin{bmatrix} K_{\mu w^{(2)}}^T & K_{z w^{(2)}}^T \end{bmatrix} \begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix} + K_{w^{(2)}}^T w^{(2)}$$

$$+ \begin{bmatrix} K_{\mu y^{(2)}}^T & K_{z y^{(2)}}^T \end{bmatrix} \begin{bmatrix} \mu y^{(2)} \\ z y^{(2)} \end{bmatrix} + K_{wy^{(2)}}^T wy^{(2)} + K_{y^{(2)}}^T y^{(2)}$$

Now we need to use equations VI.28 and VI.29, and define a few more structural matrices (which are developed more fully in Appendix C),

$$\begin{bmatrix} \mu y^{(2)} \\ z y^{(2)} \end{bmatrix}_{cm} \equiv M_6 (\Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \tag{VI.67}$$

$$\begin{bmatrix} wy^{(2)} \end{bmatrix}_{cm} \equiv M_7 (\Omega_L, \Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \tag{VI.68}$$

$$\begin{bmatrix} y^{(2)} \end{bmatrix}_{cm} \equiv M_8 (\Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \tag{VI.69}$$

From Appendix C we have $M_3(\Omega_L) = 0$, $M_4(\Omega_L, \Pi_L) = 0$, $M_5(\Omega_L, \Pi_L) = 0$, and $M_7(\Omega_L, \Pi_L) = 0$ when $\Omega_L = 0$, so we can rewrite equation VI.66 for the quadratic control input on the center manifold as

$$v^{(2)} = K_w^T \Omega_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + K_y^T \Pi_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \tag{VI.70}$$

$$+ \begin{bmatrix} K^T_{\mu^{(2)}} & K^T_{\mu z^{(2)}} & K^T_{z^{(2)}} \end{bmatrix} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix}$$

$$+ \begin{bmatrix} K^T_{\mu y^{(2)}} & K^T_{zy^{(2)}} \end{bmatrix} M_6(\Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + K^T_{y^{(2)}} M_8(\Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix}$$

where we have used the notation

$$(w)^{(2)} = \Omega_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \tag{VI.71}$$

$$(y)^{(2)} = \Pi_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \tag{VI.72}$$

Now, we can plug equation VI.70 into equation VI.65. Rearranging and again setting $M_3(\Omega_L) = 0$, $M_4(\Omega_L, \Pi_L) = 0$ and $M_5(\Omega_L, \Pi_L) = 0$, we get

$$BK^T_w \Omega_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + B \begin{bmatrix} K^T_{\mu^{(2)}} & K^T_{\mu z^{(2)}} & K^T_{z^{(2)}} \end{bmatrix} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \tag{VI.73}$$

$$+ B \begin{bmatrix} K^T_{\mu y^{(2)}} & K^T_{zy^{(2)}} \end{bmatrix} M_6(\Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix}$$

$$+ BK^T_{y^{(2)}} M_8(\Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + Q_{y_c} M_2(\Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix}$$

$$- \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{z_{P_1}} + Q_{z_{m_1}} M_1(\Pi_L) + Q_{z_c} M_2(\Pi_L) \right) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix}$$

$$
= \Pi_Q D_\xi \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} - \left(A + BK_y^T\right) \Pi_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix}
$$

which is a mess. But it is only true if

$$
\begin{aligned}
& \Pi_Q D_\xi - \left(A + BK_y^T\right) \Pi_Q \\
= \; & B \begin{bmatrix} K_{\mu^{(2)}}^T & K_{\mu z^{(2)}}^T & K_{z^{(2)}}^T \end{bmatrix} \\
+ \; & B \left(K_w^T \Omega_Q + \begin{bmatrix} K_{\mu y^{(2)}}^T & K_{zy^{(2)}}^T \end{bmatrix} M_6 \left(\Pi_L\right) + K_{y^{(2)}}^T M_8 \left(\Pi_L\right) \right) \\
+ \; & Q_{y_c} M_2 \left(\Pi_L\right) - \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{zP_1} + Q_{zm_1} M_1 \left(\Pi_L\right) + Q_{z_c} M_2 \left(\Pi_L\right) \right)
\end{aligned}
\tag{VI.74}
$$

From the theorem, we have defined

$$
\begin{aligned}
\tilde{K}_{(2)}^T &= \begin{bmatrix} K_{\mu^{(2)}}^T & K_{\mu z^{(2)}}^T & K_{z^{(2)}}^T \end{bmatrix} \\
&+ \left(K_w^T \Omega_Q + \begin{bmatrix} K_{\mu y^{(2)}}^T & K_{zy^{(2)}}^T \end{bmatrix} M_6 \left(\Pi_L\right) + K_{y^{(2)}}^T M_8 \left(\Pi_L\right) \right)
\end{aligned}
\tag{VI.75}
$$

$$
\begin{aligned}
\Gamma_z \left(\Pi_L\right) &= Q_{y_c} M_2 \left(\Pi_L\right) \\
&- \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{zP_1} + Q_{zm_1} M_1 \left(\Pi_L\right) + Q_{z_c} M_2 \left(\Pi_L\right) \right)
\end{aligned}
\tag{VI.76}
$$

which allows us to rewrite equation VI.74 more simply as

$$
\Pi_Q D_\xi - \left(A + BK_y^T\right) \Pi_Q = B\tilde{K}_{(2)}^T + \Gamma_z \left(\Pi_L\right) \tag{VI.77}
$$

As long as the gains K_y are picked so that the matrix $\left(A + BK_y^T\right)$ is strictly stable, then there is a unique solution for Π_Q by the Sylvester Equation lemma. This proves the last part of the theorem. ◁

Now we prove a corollary of the above theorem, which shows that the gains K_w^T, $K_{\mu y^{(2)}}^T$, $K_{zy^{(2)}}^T$ and $K_{y^{(2)}}^T$ may be set to zero without loss of generality.

Corollary B.4 *For any vector $\tilde{K}_{(2)}^T$ of the form*

$$
\begin{aligned}
\tilde{K}_{(2)}^T &= \begin{bmatrix} K_{\mu^{(2)}}^T & K_{\mu z^{(2)}}^T & K_{z^{(2)}}^T \end{bmatrix} \\
&+ \left(K_w^T \Omega_Q + \begin{bmatrix} K_{\mu y^{(2)}}^T & K_{zy^{(2)}}^T \end{bmatrix} M_6 \left(\Pi_L\right) + K_{y^{(2)}}^T M_8 \left(\Pi_L\right) \right)
\end{aligned}
\tag{VI.78}
$$

any desired value can be obtained by setting K_w^T, $K_{\mu y(2)}^T$, $K_{zy(2)}^T$ *and* $K_{y(2)}^T$ *to zero and choosing suitable values of* $K_{\mu(2)}^T$, $K_{\mu z(2)}^T$ *and* $K_{z(2)}^T$.

Proof. Any desired value of the generalized gain vector $\tilde{K}_{(2)}^T$ can be reached directly through the gain vector $\begin{bmatrix} K_{\mu(2)}^T & K_{\mu z(2)}^T & K_{z(2)}^T \end{bmatrix}$, without requiring input from the other possible gains in the equation VI.78. That is, even if the gains K_w^T, $K_{\mu y(2)}^T$, $K_{zy(2)}^T$ and $K_{y(2)}^T$ are set to zero, any desired value of the gain vector $\tilde{K}_{(2)}^T$ can still be reached by choosing appropriate values for the gain vector $\begin{bmatrix} K_{\mu(2)}^T & K_{\mu z(2)}^T & K_{z(2)}^T \end{bmatrix}$. Thus, the gains K_w^T, $K_{\mu y(2)}^T$, $K_{zy(2)}^T$ and $K_{y(2)}^T$ may be set to zero without loss of generality. ◁

We note that one result of this corollary is that the structural matrices $M_6(\Pi_L)$ and $M_8(\Pi_L)$ need never be calculated, and that the dynamics of the uncontrollable part of the center manifold, $\Omega(\mu, z)$, do not enter into the gains which produce the controllable part of the center manifold, $\Pi(\mu, z)$.

C. SOLVING THE CENTER MANIFOLD EQUATIONS

Now we would like to solve equations VI.45, VI.46, VI.47, and VI.48 for the linear and quadratic center manifold components, Ω_L, Π_L, Ω_Q, and Π_Q, and calculate the feedback gains which achieve these components. Since equation VI.45 gives $\Omega_L = 0$, we start by applying the results of the Unstacking Theorem from Chapter V to equation VI.46, which yields

$$\left(\begin{bmatrix} \begin{bmatrix} 0 & F_\mu^T \\ 0 & F_z^T \end{bmatrix} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \begin{bmatrix} 0 & F_\mu^T \\ 0 & F_z^T \end{bmatrix} \end{bmatrix} - \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & I \\ K_{y_1} I & & \cdots & & K_{y_p} I \end{bmatrix} \right) \begin{bmatrix} \Pi_{L_1}^T \\ \vdots \\ \Pi_{L_p}^T \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \begin{bmatrix} K_\mu \\ K_z \end{bmatrix} \end{bmatrix} \tag{VI.79}$$

where we have used the convention

$$\Pi_L = \begin{bmatrix} \Pi_{L_1} \\ \vdots \\ \Pi_{L_p} \end{bmatrix} \in R^{p\times(r+q)} \tag{VI.80}$$

and where Π_{L_i} indicates the ith row of the matrix Π_L. We can solve equation VI.79 in block form as

$$\begin{bmatrix} 0 & F_\mu^T \\ 0 & F_z^T \end{bmatrix} \Pi_{L_1}^T - \Pi_{L_2}^T = 0 \tag{VI.81}$$

$$\vdots$$

$$\begin{bmatrix} 0 & F_\mu^T \\ 0 & F_z^T \end{bmatrix} \Pi_{L_{p-1}}^T - \Pi_{L_p}^T = 0 \tag{VI.82}$$

$$\begin{bmatrix} 0 & F_\mu^T \\ 0 & F_z^T \end{bmatrix} \Pi_{L_p}^T - K_{y_1}\Pi_{L_1}^T - \ldots - K_{y_p}\Pi_{L_p}^T = \begin{bmatrix} K_\mu \\ K_z \end{bmatrix} \tag{VI.83}$$

The solution of these equations is the subject of our next theorem.

Theorem C.1 (Linear Center Manifold Solution) *The linear center manifold coefficient matrix Ω_L was previously calculated as*

$$\Omega_L = 0 \tag{VI.84}$$

The rows of the linear center manifold coefficient matrix Π_L can be calculated as linear functions of the first row of the matrix according to the formula

$$\Pi_{L_i} = \Pi_{L_1} \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^{i-1} \tag{VI.85}$$

where Π_{L_i} indicates the ith row of the matrix Π_L, and Π_{L_1} indicates the first row. Additionally the linear state feedback gains required to achieve the desired value of Π_L

are given by the formula

$$\begin{bmatrix} K_\mu^T & K_z^T \end{bmatrix} = \Pi_{L_1}\left(\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^p - \sum_{i=1}^{p} K_{y_i}\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^{i-1}\right) \tag{VI.86}$$

where we have used the definition $\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^0 = I$*, and where we have used*

$$\begin{aligned} K_\mu^T &= \begin{bmatrix} K_{\mu_1} & \cdots & K_{\mu_r} \end{bmatrix} \in R^{1\times r} && \text{(VI.87)} \\ K_z^T &= \begin{bmatrix} K_{z_1} & \cdots & K_{z_r} \end{bmatrix} \in R^{1\times q} && \text{(VI.88)} \\ K_y^T &= \begin{bmatrix} K_{y_1} & \cdots & K_{y_p} \end{bmatrix} \in R^{1\times p} && \text{(VI.89)} \end{aligned}$$

with $\mu \in R^r$*,* $z \in R^q$ *and* $y \in R^p$*.*

Proof. The previous theorem proved that $\Omega_L = 0$. To solve for Π_L, equations VI.81 through VI.82 can be solved individually and transposed to yield

$$\Pi_{L_2} = \Pi_{L_1}\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} \tag{VI.90}$$

$$\vdots$$

$$\Pi_{L_p} = \Pi_{L_{p-1}}\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} \tag{VI.91}$$

Plugging in each equation in turn from the top down yields

$$\Pi_{L_2} = \Pi_{L_1}\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} \tag{VI.92}$$

$$\vdots$$

$$\Pi_{L_p} = \Pi_{L_1}\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^{p-1} \tag{VI.93}$$

which is the desired result for the first part of the theorem. The last equation can also be transposed, and when the results of the first part of the theorem are plugged in, and rearranged, yields the desired second result of the theorem, where we have used the definition

$$\Pi_{L_1} = \Pi_{L_1} I = \Pi_{L_1}\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^0 \tag{VI.94}$$

$\triangleleft$

Now we would like to solve for Ω_Q and Π_Q. To solve for Ω_Q we apply the Unstacking Theorem of Chapter V to equation VI.47 and obtain

$$\left(\begin{bmatrix} D_\xi^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_\xi^T \end{bmatrix} - \begin{bmatrix} F_{w_{11}} I & \cdots & \cdots & F_{w_{1m}} I \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ F_{w_{m1}} I & \cdots & \cdots & F_{w_{mm}} I \end{bmatrix} \right) \begin{bmatrix} \Omega_{Q_1}^T \\ \vdots \\ \Omega_{Q_m}^T \end{bmatrix} = \begin{bmatrix} \Gamma_{w_1}^T (\Pi_L) \\ \vdots \\ \Gamma_{w_m}^T (\Pi_L) \end{bmatrix} \tag{VI.95}$$

where we have used the convention

$$\Omega_Q = \begin{bmatrix} \Omega_{Q_1} \\ \vdots \\ \Omega_{Q_m} \end{bmatrix} \in R^{m \times \frac{(r+q)(r+q+1)}{2}} \tag{VI.96}$$

where Ω_{Q_i} indicates the ith row of the matrix Ω_Q. We can solve equation VI.95 for Ω_Q by rows to yield

$$\begin{bmatrix} \Omega_{Q_1}^T \\ \vdots \\ \Omega_{Q_m}^T \end{bmatrix} = \left(\begin{bmatrix} D_\xi^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_\xi^T \end{bmatrix} - \begin{bmatrix} F_{w_{11}} I & \cdots & \cdots & F_{w_{1m}} I \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ F_{w_{m1}} I & \cdots & \cdots & F_{w_{mm}} I \end{bmatrix} \right)^{-1} \begin{bmatrix} \Gamma_{w_1}^T (\Pi_L) \\ \vdots \\ \Gamma_{w_m}^T (\Pi_L) \end{bmatrix} \tag{VI.97}$$

where we are guaranteed that the inverse exists, since Ω_Q was shown to have a unique solution in the proof of the Center Manifold theorem. Now we would like to solve for Π_Q and the quadratic state feedback gains. We apply the Unstacking Theorem of Chapter V to equation VI.48 and obtain

$$\left(\begin{bmatrix} D_\xi^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_\xi^T \end{bmatrix} - \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & I \\ K_{y_1} I & & \cdots & & K_{y_p} I \end{bmatrix} \right) \begin{bmatrix} \Pi_{Q_1}^T \\ \vdots \\ \Pi_{Q_p}^T \end{bmatrix} \tag{VI.98}$$

$$= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \tilde{K}_{(2)} \end{bmatrix} + \begin{bmatrix} \Gamma_{z_1}^T(\Pi_L) \\ \vdots \\ \Gamma_{z_p}^T(\Pi_L) \end{bmatrix}$$

where we have used the convention

$$\Pi_Q = \begin{bmatrix} \Pi_{Q_1} \\ \vdots \\ \Pi_{Q_p} \end{bmatrix} \in R^{p \times \frac{(r+q)(r+q+1)}{2}} \tag{VI.99}$$

where Π_{Q_i} indicates the ith row of the matrix Π_Q. We can solve equation VI.98 in block form as

$$D_\xi^T \Pi_{Q_1}^T - \Pi_{Q_2}^T = \Gamma_{z_1}^T(\Pi_L) \tag{VI.100}$$

$$\vdots$$

$$D_\xi^T \Pi_{Q_{p-1}}^T - \Pi_{Q_p}^T = \Gamma_{z_{p-1}}^T(\Pi_L) \tag{VI.101}$$

$$D_\xi^T \Pi_{Q_p}^T - K_{y_1}\Pi_{Q_1}^T - \ldots - K_{y_p}\Pi_{Q_p}^T = \tilde{K}_{(2)} + \Gamma_{z_p}^T(\Pi_L) \tag{VI.102}$$

The solution of these equations is the subject of our next theorem.

Theorem C.2 (Quadratic Center Manifold Solution) *The rows of the quadratic center manifold coefficient matrix Ω_Q are found by solving equation VI.97. The rows of the quadratic center manifold coefficient matrix Π_Q can be calculated as functions of the first row according to the formula*

$$\Pi_{Q_i} = \Pi_{Q_1} D_\xi^{i-1} - \sum_{j=1}^{i-1} \Gamma_{z_j}(\Pi_L) D_\xi^{i-j-1} \tag{VI.103}$$

where Π_{Q_i} indicates the ith row of the matrix Π_Q, and Π_{Q_1} indicates the first row, where we have used the definition $D_\xi^0 = I$, and where the rows of the matrix

$$\Gamma_z(\Pi_L) = \begin{bmatrix} \Gamma_{z_1}(\Pi_L) \\ \vdots \\ \Gamma_{z_p}(\Pi_L) \end{bmatrix} \in R^{p \times \frac{(r+q)(r+q+1)}{2}} \tag{VI.104}$$

are given by equation VI.51. Additionally the quadratic state feedback gains which achieve the desired value of Π_Q are given by the formula

$$\tilde{K}_{(2)}^T = \Pi_{Q_1}\left(D_\xi^p - \sum_{i=1}^{p} K_{y_i} D_\xi^{i-1}\right) + \sum_{i=1}^{p} K_{y_i} \sum_{j=1}^{i-1} \Gamma_{z_j}(\Pi_L) D_\xi^{i-j-1} - \Gamma_{z_p}(\Pi_L) \tag{VI.105}$$

The definition of $\tilde{K}_{(2)}^T$ is given in equation VI.49, but using the results of the corollary to the Center Manifold Theorem we can set the gains K_w^T, $K_{\mu y(2)}^T$, $K_{zy(2)}^T$ and $K_{y(2)}^T$ to zero, which yields a more compact formula for the quadratic gains

$$\begin{bmatrix} K_{\mu(2)}^T & K_{\mu z(2)}^T & K_{z(2)}^T \end{bmatrix} = \Pi_{Q_1} D_K + \Gamma_K^T \tag{VI.106}$$

where we have defined

$$D_K = D_\xi^p - \sum_{i=1}^{p} K_{y_i} D_\xi^{i-1} \tag{VI.107}$$

and

$$\Gamma_K^T = \sum_{i=1}^{p} K_{y_i} \sum_{j=1}^{i-1} \Gamma_{z_j}(\Pi_L) D_\xi^{i-j-1} - \Gamma_{z_p}(\Pi_L) \tag{VI.108}$$

Proof. Equations VI.100 through VI.101 can be solved individually and transposed to yield

$$\Pi_{Q_2} = \Pi_{Q_1} D_\xi - \Gamma_{z_1}(\Pi_L) \tag{VI.109}$$

$$\vdots$$

$$\Pi_{Q_p} = \Pi_{Q_{p-1}} D_\xi - \Gamma_{z_{p-1}}(\Pi_L) \tag{VI.110}$$

Plugging in each equation in turn from the top down yields

$$\Pi_{Q_2} = \Pi_{Q_1} D_\xi - \Gamma_{z_1}(\Pi_L) \tag{VI.111}$$

$$\Pi_{Q_3} = \Pi_{Q_1} D_\xi^2 - (\Gamma_{z_1}(\Pi_L) D_\xi + \Gamma_{z_2}(\Pi_L)) \tag{VI.112}$$

$$\vdots$$

$$\Pi_{Q_p} = \Pi_{Q_1} D_\xi^{p-1} - \left(\Gamma_{z_1}(\Pi_L) D_\xi^{p-2} + \ldots + \Gamma_{z_{p-2}}(\Pi_L) D_\xi + \Gamma_{z_{p-1}}(\Pi_L)\right) \tag{VI.113}$$

which can be written in index form as

$$\Pi_{Q_i} = \Pi_{Q_1} D_\xi^{i-1} - \sum_{j=1}^{i-1} \Gamma_{z_j}(\Pi_L) D_\xi^{i-j-1} \tag{VI.114}$$

which is the desired result for the first part of the theorem. Equation VI.102 can also be transposed, which gives

$$\tilde{K}_{(2)}^T = \Pi_{Q_p} D_\xi - K_{y_1} \Pi_{Q_1} - \ldots - K_{y_p} \Pi_{Q_p} - \Gamma_{z_p}(\Pi_L) \tag{VI.115}$$

which can be written in index form as

$$\tilde{K}^T_{(2)} = \Pi_{Q_p} D_\xi - \sum_{i=1}^{p} K_{y_i} \Pi_{Q_i} - \Gamma_{z_p}(\Pi_L) \tag{VI.116}$$

Plugging in equation VI.114 for Π_{Q_i} yields the desired second result of the theorem, and the results of the corollary follow directly from the definition of $\tilde{K}^T_{(2)}$. ◁

VII. STATE FEEDBACK CONTROL OF LINEARLY UNSTABILIZABLE BIFURCATIONS

A. ROADMAP: THE BIG PICTURE

1. Results of Previous Chapters

In Chapters II, III, and V we showed that any affine system

$$\dot{\breve{x}} = \breve{f}\left(\breve{x}, \breve{\mu}\right) + \breve{g}\left(\breve{x}, \breve{\mu}\right)\breve{u} \tag{VII.1}$$

can be put into quadratic normal form

$$\dot{\mu} = 0 \tag{VII.2}$$

$$\begin{aligned}
\dot{z} &= F_\mu \mu + F_z z \\
&+ Q_{z_{P_1}} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + Q_{z_{m_1}} \begin{bmatrix} \mu y_1 \\ z y_1 \end{bmatrix} + Q_{z_c} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \\
&+ Q_{z_{P_2}} \begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix} + Q_{z_{P_3}} \begin{bmatrix} w^{(2)} \end{bmatrix} + Q_{z_{m_2}} \begin{bmatrix} w y_1 \end{bmatrix} \\
&+ f_z^{(3)}\left(\mu, z, w, y\right) + g_z^{(2)}\left(\mu, z, w, y\right) v + O^{(4+)}
\end{aligned} \tag{VII.3}$$

$$\begin{aligned}
\dot{w} &= F_w w \\
&+ Q_{w_{P_1}} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + Q_{w_{m_1}} \begin{bmatrix} \mu y_1 \\ z y_1 \end{bmatrix} + Q_{w_c} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \\
&+ Q_{w_{P_2}} \begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix} + Q_{w_{P_3}} \begin{bmatrix} w^{(2)} \end{bmatrix} + Q_{w_{m_2}} \begin{bmatrix} w y_1 \end{bmatrix} + O^{(3+)}
\end{aligned} \tag{VII.4}$$

$$\dot{y} = Ay + Bv + Q_{y_c} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} + O^{(3+)} \tag{VII.5}$$

where $\mu \in R^r$ is the vector of parameters, $z \in R^q$ is the vector of linearly uncontrollable states having zero real-part eigenvalues, $w \in R^m$ is the vector of linearly uncontrollable states having non-zero real-part eigenvalues, $y \in R^p$ is the vector of linearly controllable states, $v \in R^1$ is a single control input, and A and B are the appropriately scaled Brunovsky form matrices. We can pick the control $v = v(\mu, z, w, y)$ as state feedback. It's expansion through quadratic order is

$$\begin{aligned} v &= K_\mu^T \mu + K_z^T z + K_w^T w + K_y^T y \qquad \text{(VII.6)} \\ &+ K_{\mu(2)}^T \mu^{(2)} + K_{\mu z(2)}^T \mu z^{(2)} + K_{z(2)}^T z^{(2)} + K_{\mu w(2)}^T \mu w^{(2)} + K_{zw(2)}^T z w^{(2)} + K_{w(2)}^T w^{(2)} \\ &+ K_{\mu y(2)}^T \mu y^{(2)} + K_{zy(2)}^T z y^{(2)} + K_{wy(2)}^T w y^{(2)} + K_{y(2)}^T y^{(2)} \end{aligned}$$

where the vector of gains K have yet to be chosen.

In Chapter VI we showed that the center manifold of a general affine control system in quadratic normal form could be controlled by state feedback, with linear feedback controlling the quadratic order terms of the center manifold and quadratic feedback controlling the cubic terms as a general rule. The center manifold was defined by the relations

$$w_{cm} = \Omega(\mu, z) \tag{VII.7}$$

$$y_{cm} = \Pi(\mu, z) \tag{VII.8}$$

where the notation w_{cm} and y_{cm} indicates the values of w and y on the center manifold surface. We showed that the center manifold function $\Pi(\mu, z)$ was only a function of its first component, and that the linear and quadratic Taylor series coefficient matrices were given by the formulas

$$\Pi_{L_i} = \Pi_{L_1} \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^{i-1} \tag{VII.9}$$

and

$$\Pi_{Q_i} = \Pi_{Q_1} D_\xi^{i-1} - \sum_{j=0}^{i-1} \Gamma_{z_j} \left(\Pi_L\right) D_\xi^{i-j-1} \tag{VII.10}$$

The vector of gains K can be chosen to produce a desired center manifold function $\Pi(\mu, z)$ with the formulas

$$\begin{bmatrix} K_\mu^T & K_z^T \end{bmatrix} = \Pi_{L_1} \left(\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^p - \sum_{i=1}^{p} K_{y_i} \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^{i-1} \right) \tag{VII.11}$$

and

$$\tilde{K}_{(2)}^T = \Pi_{Q_1} D_K + \Gamma_K^T \tag{VII.12}$$

where D_K and Γ_K^T are matrices defined in Chapter VI. We showed that without loss of generality, the quadratic gain vector could be reduced to

$$\tilde{K}_{(2)}^T = \begin{bmatrix} K_{\mu(2)}^T & K_{\mu z(2)}^T & K_{z(2)}^T \end{bmatrix} \tag{VII.13}$$

and that the linear gain vector K_w and all other quadratic gains can be set to zero.

2. Purpose of this Chapter

In this chapter we look at applying control of the center manifold to stabilize linearly unstabilizable bifurcations which occur in the states z of our system. In effect, in the previous chapter we focussed on the y and w dynamics. In this chapter we focus on the z dynamics.

B. THE GENERAL METHOD FOR STABILIZING SYSTEMS WITH BIFURCATIONS

The general method we will employ in the stabilization of all of the bifurcations we considered in Chapter IV and will consider in this chapter consists of the following steps:

1. Determine if a bifurcation occurs in the system of interest using the method of Chapter II. If so, trim the system to, and translate the origin of coordinates to, the equilibrium point of interest at the point of bifurcation.

2. Determine the linear properties of the trimmed system by transforming the system into linear normal form using the method of Chapter III. There are four possible cases, the first three of which were considered in Chapter IV, and the last will be considered in this chapter:

- The system is linearly controllable. That is, all states (except for the appended vector of parameters) are linearly controllable. This case was considered in Chapter IV.
- The system is linearly stabilizable. That is, all linearly uncontrollable states (except for the appended vector of parameters) have eigenvalues with negative real parts and are exponentially stable. This case was considered in Chapter IV.
- The system is linearly unstable. That is, at least one linearly uncontrollable state has an eigenvalue with a positive real part and is exponentially unstable. This case was considered in Chapter IV.
- The system is linearly unstabilizable, but not linearly unstable. That is, all linearly uncontrollable states have eigenvalues with either negative or zero real parts, but there are no eigenvalues with positive real parts. This is the case we will consider in this chapter.

3. Stabilize the linearly controllable states with linear state feedback as described in Chapter IV. Any linear control method (pole placement, linear quadratic regulator, robust control, etc.) which produces a stable closed loop matrix $A + BK_y^T$ is acceptable.

4. If the system is strictly linearly unstable, then stabilization is not possible using the methods in this dissertation.

5. If the system is linearly unstabilizable, but not strictly linearly unstable, then stability depends on the non-linear terms. Determine the underlying dynamics of the states with linearly uncontrollable, zero real part eigenvalues by transforming the system into quadratic normal form using the method of Chapter V. This will determine the type of bifurcation in the linearly uncontrollable states which needs to be controlled. Then apply the appropriate control laws using the methods of this chapter. These can be summarized as:

- Stabilize the linearly controllable states with linear state feedback as previously discussed.
- Determine the desired quadratic order dynamics for the linearly uncontrollable, zero real part eigenvalue states after control has been applied. This will allow calculation of the linear coefficients of the center manifold, which will allow calculation of the linear feedback gains for these states.
- Determine the desired cubic order dynamics for the linearly uncontrollable, zero real part eigenvalue states after control has been applied. This will allow calculation of the quadratic coefficients of the center manifold, which will allow calculation of the quadratic feedback gains for these states.

6. Transform the stabilizing feedback into the original system by reversing all translations and transformations used to put the system into normal form, as described at the end of Chapter V.

C. LINEARLY UNSTABILIZABLE BIFURCATIONS

A bifurcation is linearly unstabilizable if, after the system is transformed into linear normal form, states z exist which are linearly unstabilizable but not linearly unstable, that is, the system has the form

$$\dot{\mu} = 0 \tag{VII.14}$$

$$\dot{z} = F_\mu \mu + F_z z + O^{(2+)} \tag{VII.15}$$

$$\dot{w} = F_w w + O^{(2+)} \tag{VII.16}$$

$$\dot{y} = Ay + Bv + O^{(2+)} \tag{VII.17}$$

where the matrices F_μ, F_z, F_w, A, and B are in Jordan-Brunovsky canonical form, and all the eigenvalues of F_w have negative real parts. (In certain systems of this form, the vector w (and by implication the matrix F_w) may not exist. That is, all the linearly uncontrollable states may have eigenvalues with zero real parts. In these cases, deleting all reference to the states w in the general equations gives the right

answer.) Since linear stabilization of the states z involved in the bifurcation is not possible, in general we will not be able control these states. However, we would still like to be able to use our control to affect the dynamics of these states in a favorable manner. To investigate our ability to affect these states through the non-linear terms, we transform our system into quadratic normal form using the method of Chapter V. The general quadratic normal form is given in equations VII.2, VII.3, VII.4, and VII.5. Since linear control or stabilization of the bifurcation is not possible, our non-linear control strategy will instead be:

- To stabilize the bifurcation if stabilization is possible, which normally means that after control is applied, the linearly uncontrollable states are attracted to a stable (but not necessarily linearly stable) equilibrium point, which, however, may shift location as the parameter is varied;
- If stabilization is not possible, to soften the bifurcation if softening is possible, which normally means that the bifurcation is converted to a supercritical form with desired coefficients;
- If softening is not possible, to know that the bifurcation is not controllable, stabilizable, or softenable in any of the above senses.

Now we look at the effect of the shape of the center manifold on the dynamics of the linearly uncontrollable states.

1. General Center Manifold Dynamics

Certain characteristics of the dynamics on the center manifold depend on the shape of the center manifold, and are the same for all linearly unstabilizable bifurcations, which we will develop in this section. In later sections we will look at how to manipulate the shape of the center manifold to achieve the dynamics desired for each individual type of bifurcation. For a general system in normal form exhibiting a bifurcation, the dynamics on the center manifold are given by equation VII.3, which

we rewrite here for convenience as

$$\begin{aligned}
\dot{z} &= F_\mu \mu + F_z z + Q_{zP_1} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + Q_{zm_1} \begin{bmatrix} \mu y_1 \\ z y_1 \end{bmatrix} + Q_{zc} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \qquad \text{(VII.18)} \\
&+ Q_{zP_2} \begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix} + Q_{zP_3} \begin{bmatrix} w^{(2)} \end{bmatrix} + Q_{zm_2} \begin{bmatrix} w y_1 \end{bmatrix} \\
&+ f_z^{(3)}(\mu, z, w, y) + g_z^{(2)}(\mu, z, w, y)\, v + O^{(4+)}
\end{aligned}$$

where we have $\mu \in R^r$, $z \in R^q$, $w \in R^m$, $y \in R^p$ and where

$$F_\mu \in R^{q\times r} \qquad \text{(VII.19)}$$

$$F_z \in R^{q\times q} \qquad \text{(VII.20)}$$

$$Q_{zP_1} \in R^{q\times \frac{(r+q)(r+q+1)}{2}} \qquad \text{(VII.21)}$$

$$Q_{zm_1} \in R^{q\times (r+q)} \qquad \text{(VII.22)}$$

$$Q_{zc} \in R^{q\times p} \qquad \text{(VII.23)}$$

$$Q_{zP_2} \in R^{q\times (r+q)m} \qquad \text{(VII.24)}$$

$$Q_{zP_3} \in R^{q\times \frac{(m)(m+1)}{2}} \qquad \text{(VII.25)}$$

$$Q_{zm_2} \in R^{q\times m} \qquad \text{(VII.26)}$$

are matrices of constant coefficients, which take on different values depending on the type of bifurcation involved, as set out in Appendix D.

Now we want to determine what happens to our z dynamics given by equation VII.18 when y and w have collapsed to the center manifold. On the center manifold, from Chapter VI, we have

$$w_{cm} = \Omega_L \begin{bmatrix} \mu \\ z \end{bmatrix} + \Omega_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \qquad \text{(VII.27)}$$

and

$$y_{cm} = \Pi_L \begin{bmatrix} \mu \\ z \end{bmatrix} + \Pi_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \tag{VII.28}$$

From Chapter VI we also have

$$\Omega_L = 0 \tag{VII.29}$$

and that

$$\Omega_L \in R^{m\times(r+q)} \tag{VII.30}$$

$$\Omega_Q \in R^{m\times\frac{(r+q)(r+q+1)}{2}} \tag{VII.31}$$

$$\Pi_L \in R^{p\times(r+q)} \tag{VII.32}$$

$$\Pi_Q \in R^{p\times\frac{(r+q)(r+q+1)}{2}} \tag{VII.33}$$

are matrices of constant coefficients, which we will be determining as part of the control problem. From Appendix C we have

$$\begin{bmatrix} \mu y_1 \\ z y_1 \end{bmatrix}_{cm} = M_1(\Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_1(\Pi_Q) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{VII.34}$$

$$\begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix}_{cm} = M_2(\Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_2(\Pi_L, \Pi_Q) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{VII.35}$$

$$\begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix}_{cm} = N_3(\Omega_Q) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{VII.36}$$

$$\left[w^{(2)}\right]_{cm} = 0 + O^{(4+)} \tag{VII.37}$$

$$\left[\, wy_1 \,\right]_{cm} = N_5\left(\Pi_L, \Omega_Q\right) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{VII.38}$$

where we have included the fact that $M_3 = M_4 = M_5 = 0$, and that $N_4 = 0$. We note from Chapter VI the fact that Ω_Q is only a function of Π_L, which implies that $N_3(\Omega_Q)$ and $N_4(\Pi_L, \Omega_Q)$ are also functions only of Π_L (see Appendix C for discussion). Plugging equations VII.34, VII.35, VII.36, VII.37 and VII.38 into equation VII.18, we have the equation for the dynamics on the center manifold

$$\begin{aligned} \dot{z} &= F_\mu \mu + F_z z + \left(Q_{z_{P_1}} + Q_{z_{m_1}} M_1\left(\Pi_L\right) + Q_{z_c} M_2\left(\Pi_L\right)\right) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \qquad \text{(VII.39)} \\ &+ \left(Q_{z_{m_1}} N_1\left(\Pi_Q\right) + Q_{z_c} N_2\left(\Pi_L, \Pi_Q\right)\right) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} \\ &+ \left(Q_{z_{P_2}} N_3\left(\Omega_Q\right) + Q_{z_{m_2}} N_5\left(\Pi_L, \Omega_Q\right)\right) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} \\ &+ f_z^{(3)}\left(\mu, z, w_{cm}^{(1)}, y_{cm}^{(1)}\right) + g_z^{(2)}\left(\mu, z, w_{cm}^{(1)}, y_{cm}^{(1)}\right) \dot{v}^{(1)} + O^{(4+)} \end{aligned}$$

Now we rewrite equation VII.39 in the form of a theorem.

Theorem C.1 (General Center Manifold Control) *The center manifold dynamics of a system in the form of equation VII.18 are given by the equation*

$$\dot{z} = F_\mu \mu + F_z z + Q_{cm}\left(\Pi_L\right) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + C_{cm}\left(\Pi_L, \Pi_Q\right) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{VII.40}$$

where the matrices $Q_{cm}(\Pi_L)$ and $C_{cm}(\Pi_L, \Pi_Q)$ are given by

$$Q_{cm}\left(\Pi_L\right) = Q_{z_{P_1}} + Q_{z_{m_1}} M_1\left(\Pi_L\right) + Q_{z_c} M_2\left(\Pi_L\right) \tag{VII.41}$$

$$C_{cm}\left(\Pi_L, \Pi_Q\right) = \tilde{C}_z\left(\Pi_L\right) + Q_{z_{m_1}} N_1\left(\Pi_Q\right) + Q_{z_c} N_2\left(\Pi_L, \Pi_Q\right) \tag{VII.42}$$

with the matrix $\tilde{C}_z(\Pi_L)$ *defined by the relation*

$$\tilde{C}_z(\Pi_L)\begin{bmatrix}\mu^{(3)}\\ \mu z^{(3)}\\ z^{(3)}\end{bmatrix} = \left(Q_{zP_2}N_3(\Omega_Q) + Q_{zm_2}N_5(\Pi_L,\Omega_Q)\right)\begin{bmatrix}\mu^{(3)}\\ \mu z^{(3)}\\ z^{(3)}\end{bmatrix} \qquad \text{(VII.43)}$$
$$+f_z^{(3)}\left(\mu, z, w_{cm}^{(1)}, y_{cm}^{(1)}\right) + g_z^{(2)}\left(\mu, z, w_{cm}^{(1)}, y_{cm}^{(1)}\right)v^{(1)}$$

and where the matrices M_1, M_2, N_1, N_2, N_3, *and* N_5 *are determined from Appendix C for the specific system being analyzed.*

Proof. Looking at equation VII.39, grouping all the quadratic terms gives $Q_{cm}(\Pi_L)$, and grouping all the cubic terms gives $C_{cm}(\Pi_L,\Pi_Q)$. Grouping all the cubic terms which do not depend on Π_Q gives $\tilde{C}_z(\Pi_L)$, noting that

$$\begin{aligned} v^{(1)} &= \begin{bmatrix}K_\mu^T & K_z^T\end{bmatrix}\begin{bmatrix}\mu\\ z\end{bmatrix} + K_w^T w_{cm}^{(1)} + K_y^T y_{cm}^{(1)} \qquad \text{(VII.44)}\\ &= \begin{bmatrix}K_\mu^T & K_z^T\end{bmatrix}\begin{bmatrix}\mu\\ z\end{bmatrix} + K_w^T\Omega_L\begin{bmatrix}\mu\\ z\end{bmatrix} + K_y^T\Pi_L\begin{bmatrix}\mu\\ z\end{bmatrix}\\ &= \left(\begin{bmatrix}K_\mu^T & K_z^T\end{bmatrix} + K_y^T\Pi_L\right)\begin{bmatrix}\mu\\ z\end{bmatrix}\end{aligned}$$

is only a function of Π_L, and where we have set $\Omega_L = 0$ by the Linear Center Manifold Solution theorem in Chapter VI. From the same theorem, we have

$$\begin{aligned} K_y^T\Pi_L &= \sum_{i=1}^{p} K_{y_i}\Pi_{L_i} \qquad \text{(VII.45)}\\ &= \Pi_{L_1}\sum_{i=1}^{p} K_{y_i}\begin{bmatrix}0 & 0\\ F_\mu & F_z\end{bmatrix}^{i-1}\end{aligned}$$

and

$$\begin{bmatrix}K_\mu^T & K_z^T\end{bmatrix} = \Pi_{L_1}\left(\begin{bmatrix}0 & 0\\ F_\mu & F_z\end{bmatrix}^p - \sum_{i=1}^{p} K_{y_i}\begin{bmatrix}0 & 0\\ F_\mu & F_z\end{bmatrix}^{i-1}\right) \qquad \text{(VII.46)}$$

which combined give

$$v^{(1)} = \Pi_{L_1}\begin{bmatrix}0 & 0\\ F_\mu & F_z\end{bmatrix}^p\begin{bmatrix}\mu\\ z\end{bmatrix} \qquad \text{(VII.47)}$$

Since $\Omega_L = 0$, $f_z^{(3)}\left(\mu, z, w_{cm}^{(1)}, y_{cm}^{(1)}\right)$ and $g_z^{(2)}\left(\mu, z, w_{cm}^{(1)}, y_{cm}^{(1)}\right)$ are only functions of Π_L, which means that $\tilde{C}_z$ is only a function of Π_L as stated, which completes the proof. $\triangleleft$

Now our theorem has an important consequence. By choosing the elements of Π_L we affect the quadratic and higher order coefficients of our center manifold dynamics. Then, knowing Π_L and choosing the elements of Π_Q, we affect the cubic and higher order coefficients of our center manifold dynamics. Finding the formulas for Π_L which produce the desired quadratic coefficients, and the formulas for Π_Q which produce the desired cubic coefficients, and then turning them into appropriate state feedback gains using the formulas in Chapter VI, will be our general control strategy for linearly unstabilizable bifurcations.

2. General Considerations for One Dimensional Bifurcations

In this section, we will look at the specific ways to stabilize various bifurcations occurring in one dimension, having one parameter (so-called "co-dimension one" bifurcations). The generic one dimensional co-dimension one bifurcation is a saddle-node bifurcation, which we will consider in detail in this dissertation. Also well known are the one-dimensional co-dimension one degeneracies: transcritical bifurcations, the two types of pitchfork bifurcations, and the various forms of degenerate transcritical cases, such as the isolated equilibrium point. Because these degenerate cases are special cases of the generic one dimensional case, no attempt will be made to treat them comprehensively. Instead, they will be treated as isolated special cases and dealt with on a case by case basis. For all the one dimensional bifurcations, the quadratic terms in the center manifold dynamics are inherently de-stabilizing, so the focus of our efforts at stabilization will be directed toward exactly cancelling them with the application of state feedback. Note that although exact cancellation of a term may be possible in certain cases in a mathematical sense, in actual practice this will not occur (there will always be a small error). The cubic terms, however, can be either stabilizing or de-stabilizing, depending on the sign of their coefficient. (In

a one dimensional dynamic system, a negative cubic coefficient is stabilizing, while a positive cubic coefficient is de-stabilizing.) So the second stage of our stabilization efforts will be devoted to controlling the sign and magnitude of the coefficient of the cubic term. In a practical sense then, our attempt here is to create quadratic coefficients which are sufficiently small, and cubic coefficients which are sufficiently large, such that the hysteresis created by errors in cancellation will be small in magnitude.

For a system exhibiting a one dimensional, co-dimension one bifurcation, the quadratic normal form of the z dynamics is

$$\begin{aligned} \dot{z}_1 &= F_\mu \mu_1 + Q_{zP_1} \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \end{bmatrix} + Q_{zm_1} \begin{bmatrix} \mu_1 y_1 \\ z_1 y_1 \end{bmatrix} + Q_{zc} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \\ &+ Q_{zP_2} \begin{bmatrix} \mu_1 w \\ z_1 w \end{bmatrix} + Q_{zP_3} \begin{bmatrix} w^{(2)} \end{bmatrix} + Q_{zm_2} \begin{bmatrix} w y_1 \end{bmatrix} \\ &+ f_z^{(3)}(\mu_1, z_1, w, y) + g_z^{(2)}(\mu_1, z_1, w, y)\, v + O^{(4+)} \end{aligned} \tag{VII.48}$$

and the dynamics on the center manifold are given by

$$\dot{z}_1 = F_\mu \mu_1 + Q_{cm}(\Pi_L) \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \end{bmatrix} + C_{cm}(\Pi_L, \Pi_Q) \begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1 z_1^2 \\ z_1^3 \end{bmatrix} + O^{(4+)} \tag{VII.49}$$

where we have used $\mu = [\mu_1] \in R^1$, $z = [z_1] \in R^1$, $w \in R^m$ and $y \in R^p$. The matrix F_μ is a scalar, and is non-zero for a saddle-node bifurcation, but is zero for a transcritical or pitchfork bifurcation, or any other of the one dimensional degenerate cases. Also, the matrix F_z is a scalar zero for any one dimensional bifurcation, since that is the only possible one dimensional matrix having an eigenvalue with a zero real part. (We will see in the case of two dimensional bifurcations that there are three possible choices for F_z.) $F_z = 0$ is implicit in equations VII.48 and VII.49 since it

doesn't show up. The matrices of coefficients in equation VII.48 are given by

$$Q_{zP_1} = \begin{bmatrix} q_{zP_1} & q_{zP_2} & q_{zP_3} \end{bmatrix} \tag{VII.50}$$

$$Q_{zm_1} = \begin{bmatrix} q_{zm_1} & q_{zm_2} \end{bmatrix} \tag{VII.51}$$

$$Q_{zc} = \begin{bmatrix} q_{zc_1} & q_{zc_2} & \cdots & q_{zc_p} \end{bmatrix} \tag{VII.52}$$

The elements of the matrix Q_{zP_1} take on different values depending on the type of bifurcation, as set out in Appendix D. Since the matrices Q_{zP_2}, Q_{zP_3} and Q_{zm_2} are rather complicated in general, and have only a limited effect on the outcome, we will detail them later.

Now we want to calculate the matrix of quadratic coefficients $Q_{cm}(\Pi_L)$ in equation VII.49. First we need to state the one dimensional form of the matrix Π_L, which is

$$\Pi_L = \begin{bmatrix} \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} \\ \vdots & \vdots \\ \Pi_{L_{p\mu_1}} & \Pi_{L_{pz_1}} \end{bmatrix} \tag{VII.53}$$

Next, we need to get the matrices M_1, and M_2 for a one dimensional co-dimension one bifurcation from Appendix C, which are

$$M_1(\Pi_L) = \begin{bmatrix} \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} & 0 \\ 0 & \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} \end{bmatrix} \tag{VII.54}$$

and

$$M_2(\Pi_L) = \begin{bmatrix} \left(\Pi_{L_{1\mu_1}}\right)^2 & 2\Pi_{L_{1\mu_1}}\Pi_{L_{1z_1}} & \left(\Pi_{L_{1z_1}}\right)^2 \\ \left(F_\mu \Pi_{L_{1z_1}}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \tag{VII.55}$$

Rewriting equation VII.41 for convenience, we have

$$Q_{cm}(\Pi_L) = Q_{zP_1} + Q_{zm_1} M_1(\Pi_L) + Q_{zc} M_2(\Pi_L) \tag{VII.56}$$

Plugging equations VII.50, VII.51, VII.52, C.16 and VII.55 into equation VII.56 for a one dimensional system gives

$$Q_{cm}(\Pi_L) = \begin{bmatrix} q_{cm_1} & q_{cm_2} & q_{cm_3} \end{bmatrix} \tag{VII.57}$$

with

$$q_{cm_1} = q_{zP_1} + q_{zm_1}\Pi_{L_{1\mu_1}} + q_{zc_1}\left(\Pi_{L_{1\mu_1}}\right)^2 + q_{zc_2}F_\mu^2\left(\Pi_{L_{1z_1}}\right)^2 \tag{VII.58}$$

$$q_{cm_2} = q_{zP_2} + q_{zm_2}\Pi_{L_{1\mu_1}} + q_{zm_1}\Pi_{L_{1z_1}} + 2q_{zc_1}\Pi_{L_{1\mu_1}}\Pi_{L_{1z_1}} \tag{VII.59}$$

$$q_{cm_3} = q_{zP_3} + q_{zm_2}\Pi_{L_{1z_1}} + q_{zc_1}\left(\Pi_{L_{1z_1}}\right)^2 \tag{VII.60}$$

Now, based on the Poincare normal forms attainable for each individual bifurcation (see Appendix D), we will use equations VII.58, VII.59 and VII.60 to pick the linear coefficients $\Pi_{L_{1\mu_1}}$ and $\Pi_{L_{1z_1}}$ of the y component of the center manifold surface. (This process will become clear in subsequent sections as we work out the process for the individual types of bifurcations.)

Now we want to calculate the matrix of cubic coefficients $C_{cm}(\Pi_L, \Pi_Q)$. First we need to state the one dimensional form of the matrix Π_Q, which is

$$\Pi_Q = \begin{bmatrix} \Pi_{Q_{1\mu_1^2}} & \Pi_{Q_{1\mu_1z_1}} & \Pi_{Q_{1z_1^2}} \\ \vdots & \vdots & \vdots \\ \Pi_{Q_{p\mu_1^2}} & \Pi_{Q_{p\mu_1z_1}} & \Pi_{Q_{pz_1^2}} \end{bmatrix} \tag{VII.61}$$

Next, we need to get the matrices N_1, and N_2 for a one dimensional bifurcation from Appendix C, which are

$$N_1(\Pi_Q) = \begin{bmatrix} \Pi_{Q_{1\mu_1^2}} & \Pi_{Q_{1\mu_1z_1}} & \Pi_{Q_{1z_1^2}} & 0 \\ 0 & \Pi_{Q_{1\mu_1^2}} & \Pi_{Q_{1\mu_1z_1}} & \Pi_{Q_{1z_1^2}} \end{bmatrix} \tag{VII.62}$$

and

$$N_2(\Pi_L, \Pi_Q) = \begin{bmatrix} 2\Pi_{L_{1\mu_1}}\Pi_{Q_{1\mu_1^2}} & 2\left(\Pi_{L_{1\mu_1}}\Pi_{Q_{1\mu_1 z_1}} + \Pi_{L_{1z_1}}\Pi_{Q_{1\mu_1^2}}\right) & 2\left(\Pi_{L_{1\mu_1}}\Pi_{Q_{1z_1^2}} + \Pi_{L_{1z_1}}\Pi_{Q_{1\mu_1 z_1}}\right) & 2\Pi_{L_{1z_1}}\Pi_{Q_{1z_1^2}} \\ 2F_\mu\Pi_{L_{1z_1}}\Pi_{Q_{2\mu_1^2}} & 2F_\mu\Pi_{L_{1z_1}}\Pi_{Q_{2\mu_1 z_1}} & 2F_\mu\Pi_{L_{1z_1}}\Pi_{Q_{2z_1^2}} & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{VII.63}$$

Then, we need to state the one dimensional form of the matrix $\tilde{C}_z$, which is

$$\tilde{C}_z(\Pi_L) = \begin{bmatrix} \tilde{c}_{z_1} & \tilde{c}_{z_2} & \tilde{c}_{z_3} & \tilde{c}_{z_4} \end{bmatrix} \tag{VII.64}$$

where the coefficient matrix $\tilde{C}_z$ is defined by the one dimensional ($z = [z_1]$), co-dimension one ($\mu = [\mu_1]$) case for equation VII.43, given by

$$\begin{aligned} \tilde{C}_z(\Pi_L)\begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1 z_1^2 \\ z_1^3 \end{bmatrix} &= \left(Q_{z_{P_2}} N_3(\Omega_Q) + Q_{z_{m_2}} N_5(\Pi_L, \Omega_Q)\right)\begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1 z_1^2 \\ z_1^3 \end{bmatrix} \\ &+ f_z^{(3)}\left(\mu_1, z_1, w_{cm}^{(1)}, y_{cm}^{(1)}\right) + g_z^{(2)}\left(\mu_1, z_1, w_{cm}^{(1)}, y_{cm}^{(1)}\right)v^{(1)} + O^{(4+)} \end{aligned} \tag{VII.65}$$

which is most easily calculated on a case by case basis for a given individual system. Here we will assume $\tilde{C}_z$ has already been calculated. Plugging equations C.17, VII.63, VII.64, VII.51 and VII.52 into into equation VII.42 we get

$$C_{cm}(\Pi_L, \Pi_Q) = \begin{bmatrix} c_{cm_1} & c_{cm_2} & c_{cm_3} & c_{cm_4} \end{bmatrix} \tag{VII.66}$$

where

$$c_{cm_1} = \tilde{c}_{z_1} + q_\alpha \Pi_{Q_{1\mu_1^2}} + q_\gamma \Pi_{Q_{2\mu_1^2}} \tag{VII.67}$$

$$c_{cm_2} = \tilde{c}_{z_2} + q_\beta \Pi_{Q_{1\mu_1^2}} + q_\alpha \Pi_{Q_{1\mu_1 z_1}} + q_\gamma \Pi_{Q_{2\mu_1 z_1}} \tag{VII.68}$$

$$c_{cm_3} = \tilde{c}_{z_3} + q_\beta \Pi_{Q_{1\mu_1 z_1}} + q_\alpha \Pi_{Q_{1z_1^2}} + q_\gamma \Pi_{Q_{2z_1^2}} \tag{VII.69}$$

$$c_{cm_4} = \tilde{c}_{z_4} + q_\beta \Pi_{Q_{1z_1^2}} \tag{VII.70}$$

with

$$q_\alpha = q_{zm_1} + 2q_{zc_1} \Pi_{L_{1\mu_1}} \tag{VII.71}$$

$$q_\beta = q_{zm_2} + 2q_{zc_1} \Pi_{L_{1z_1}} \tag{VII.72}$$

$$q_\gamma = 2q_{zc_2} F_\mu \Pi_{L_{1z_1}} \tag{VII.73}$$

Now, based on the cubic Poincare normal forms attainable for each individual bifurcation, we will use equations VII.67 through VII.70 to pick the quadratic coefficients $\Pi_{Q_{1\mu_1^2}}$, $\Pi_{Q_{1\mu_1 z_1}}$ and $\Pi_{Q_{1z_1^2}}$ of the y component of the center manifold surface. The quadratic coefficients $\Pi_{Q_{2\mu_1^2}}$, $\Pi_{Q_{2\mu_1 z_1}}$ and $\Pi_{Q_{2z_1^2}}$ are functions of $\Pi_{Q_{1\mu_1^2}}$, $\Pi_{Q_{1\mu_1 z_1}}$ and $\Pi_{Q_{1z_1^2}}$, as given in equation VII.10. (This process will become clear in subsequent sections as we work out the process for the individual types of bifurcations.)

Finally, we prove a specialized lemma and corollary for the general one dimensional co-dimension one case, which we need in considering the individual types of bifurcations. In Chapter VI, in the Center Manifold Theorem, we defined the matrix $\Gamma_z(\Pi_L)$ as

$$\begin{aligned} \Gamma_z(\Pi_L) &= Q_{y_c} M_2(\Pi_L) \\ &- \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{zP_1} + Q_{zm_1} M_1(\Pi_L) + Q_{zc} M_2(\Pi_L) \right) \end{aligned} \tag{VII.74}$$

where the individual rows $\Gamma_{z_i}(\Pi_L)$ are given by

$$\Gamma_z(\Pi_L) = \begin{bmatrix} \Gamma_{z_1}(\Pi_L) \\ \vdots \\ \Gamma_{z_p}(\Pi_L) \end{bmatrix} \tag{VII.75}$$

The rows of $\Gamma_z(\Pi_L)$ were then used in the Quadratic Center Manifold Solution theorem to calculate the matrix Γ_K^T used in the solution of the quadratic state feedback gains, which was defined as

$$\Gamma_K^T = -\sum_{j=0}^{p} \Gamma_{z_j}(\Pi_L) D_\xi^{p-j} + \sum_{i=1}^{p} K_{y_i} \sum_{j=0}^{i-1} \Gamma_{z_j}(\Pi_L) D_\xi^{i-j-1} \tag{VII.76}$$

where we have used the definitions $D_\xi^0 = I$ and $\Gamma_{z_0}(\Pi_L) = 0$. So, we would like to calculate the matrices $\Gamma_z(\Pi_L)$ and Γ_K^T for the general one dimensional co-dimension one case. We state our results in a lemma and a corollary.

Lemma C.2 (One Dimensional Γ_z Matrix) *For the general one dimensional, co-dimension one case, the rows of the matrix $\Gamma_z(\Pi_L)$ in equation VII.75 are given by the formulas*

$$\Gamma_{z_1}(\Pi_L) = \Pi_{L_{1_{z1}}} \left(Q_{zP_1} + Q_{zm_1} M_1(\Pi_L) + Q_{z_c} M_2(\Pi_L)\right) \tag{VII.77}$$

and

$$\Gamma_{z_i}(\Pi_L) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \tag{VII.78}$$

for $i = 2$ to p.

Proof. We wish to solve equation VII.75 by plugging terms into equation VII.74. Looking at the term $Q_{y_c} M_2(\Pi_L)$, we have $Q_{y_c} \in R^{p\times p}$ from Kang's Theorem in Chapter V given by the upper triangular form with zeros on the main and first super diagonals

$$Q_{y_c} = \begin{bmatrix} 0 & 0 & \gamma_{13} & \cdots & \gamma_{1p} \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \gamma_{(p-2)p} \\ & & & \ddots & 0 \\ 0 & & \cdots & & 0 \end{bmatrix} \tag{VII.79}$$

and we have $M_2(\Pi_L) \in R^{p\times 3}$ from Appendix C given by

$$M_2(\Pi_L) = \begin{bmatrix} \left(\Pi_{L_{1_{\mu_1}}}\right)^2 & 2\left(\Pi_{L_{1_{\mu_1}}}\right)\left(\Pi_{L_{1_{z_1}}}\right) & \left(\Pi_{L_{1_{z_1}}}\right)^2 \\ \left(F_\mu \Pi_{L_{1_{z_1}}}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \tag{VII.80}$$

When we multiply them out, because of the fact that the first two columns of Q_{y_c} are zero and because only the first two rows of $M_2(\Pi_L)$ are non-zero, we get $Q_{y_c}M_2(\Pi_L) = 0$. Next, look at the term $\Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix}$. Because $z \in R^1$, we have $I = [1]$. From Chapter VI, in the Linear Center Manifold Solution theorem, the rows of the matrix Π_L are given by

$$\Pi_{L_i} = \Pi_{L_1} \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^{i-1} \tag{VII.81}$$

where we have used the definition $\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^0 = I$. Since $F_z = 0$ for a one dimensional bifurcation, we obtain

$$\Pi_L \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \Pi_{L_{1_{z_1}}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{VII.82}$$

which yields equations VII.77 and VII.78 when plugged into equation VII.74, proving the lemma. ◁

Corollary C.3 (One Dimensional Γ_K Matrix) *For the general one dimensional, co-dimension one case, the matrix Γ_K^T is given by the formula*

$$\Gamma_K^T = -\Gamma_{z_1}(\Pi_L) \tag{VII.83}$$

for $p = 1$, and

$$\Gamma_K^T = \Gamma_{z_1}(\Pi_L)\left(-D_\xi^{p-1} + \sum_{i=2}^{p} K_{y_i} D_\xi^{i-2}\right) \tag{VII.84}$$

for $p \geq 2$.

Proof. We focus on the fact that only the row $\Gamma_{z_1}(\Pi_L)$ is non-zero for the one-dimensional case. The case where $p = 1$ is obtained by plugging the definition $\Gamma_{z_0}(\Pi_L) = 0$ into equation VII.76, which eliminates the second term entirely, and using the definition $D_\xi^0 = I$ which simplifies the first term. The case for $p \geq 2$ is obtained by plugging the above definitions and $\Gamma_{z_i}(\Pi_L) = 0$ for $i = 2$ to p from the preceding lemma into equation VII.76. ◁

We now look at the individual one dimensional bifurcations: saddle-node, transcritical and both types of pitchfork.

3. Saddle-Node Bifurcations

Saddle-node bifurcations are characterized by equations of the form

$$\dot{z}_1 = F_\mu \mu_1 + Q_z z_1^2 + C_z z_1^3 + O^{(4+)} \tag{VII.85}$$

where $\mu_1 \in R^1$, $z_1 \in R^1$, Q_z and C_z are scalar constants, and F_μ is a non-zero scalar constant. When $Q_z \neq 0$, equation VII.85 can be expressed as

$$\dot{z}_1 = F_\mu \mu_1 + Q_z z_1^2 + O^{(3+)} \tag{VII.86}$$

Analyzing the system in equation VII.85, for $Q_z \neq 0$ as in equation VII.86, the local dynamics near the origin are dominated by the quadratic term z_1^2. For $\mu_1 < 0$, there are two equilibrium points, one at

$$z_+^* = \sqrt{\frac{-F_\mu \mu_1}{Q_z}} \tag{VII.87}$$

and one at

$$z_-^* = -\sqrt{\frac{-F_\mu \mu_1}{Q_z}} \tag{VII.88}$$

while for $\mu_1 = 0$, the only equilibrium point occurs at $z^* = 0$, and for $\mu_1 > 0$, there are no equilibrium points at all. (For convenience, we assume that $F_\mu > 0$ and $Q_z > 0$. The analysis is similar for other cases.) Looking at the stability of our equilibrium

points (when they exist) by examining the eigenvalues of the Jacobian matrix of our system, gives

$$J = \left(\frac{\partial f}{\partial z_1}\right)_{z_1 = z^*} \tag{VII.89}$$

where J is the Jacobian matrix, and where

$$f(z_1, \mu_1) = F_\mu \mu_1 + Q_z z_1^2 + O^{(3+)} \tag{VII.90}$$

which gives

$$\begin{aligned} J &= 2Q_z z^* + O^{(2+)} \\ &= \pm 2Q_z \sqrt{\frac{-F_\mu \mu_1}{Q_z}} + H.O.T. \end{aligned} \tag{VII.91}$$

where we have plugged in equations VII.87 and VII.88 for z^*. For small values of μ_1 (i.e. a local bifurcation around the origin), the higher order terms may be neglected, and the initial term determines the sign of J. Since our Jacobian matrix is one-dimensional, the eigenvalue is just the value of J itself, and we can see that the equilibrium point at z_-^* is stable, while the equilibrium point at z_+^* is unstable. So, our local system dynamics can be summarized as follows: For $\mu_1 < 0$, points in the region $z_1 < z_+^*$ are attracted to z_-^* and converge, but points in the region $z_+^* < z_1$ are repelled from z_+^* and diverge; for $\mu_1 = 0$, points in the region $z_1 < 0$ are attracted to the origin and converge, but points in the region $z_1 > 0$ are repelled from the origin and diverge; and for $\mu_1 > 0$, no equilibrium points exist and all local values of z_1 diverge. So clearly, the quadratic term is destabilizing in all cases.

What about the case when no quadratic term exists? Analyzing the system in equation VII.85 for $Q_z = 0$, we have

$$\dot{z}_1 = F_\mu \mu_1 + C_z z_1^3 + O^{(4+)} \tag{VII.92}$$

and the local dynamics near the origin are dominated by the cubic term z_1^3. For any value of μ_1, there is only one equilibrium point for this case, at

$$z^* = \sqrt[3]{\frac{-F_\mu \mu_1}{C_z}} \tag{VII.93}$$

Examining the eigenvalues of the Jacobian of this system, we have

$$J = 3C_z z^{*2} \tag{VII.94}$$

and the stability of our system is determined purely by the sign of C_z. (For $C_z < 0$ the system is stable, while for $C_z > 0$ the system is unstable.) So, our control strategy in subsequent sections for a system which exhibits a saddle-node bifurcation will be to attempt to cancel the quadratic term with linear state feedback, and to produce a negative cubic term with quadratic state feedback.

a. Desired Closed Loop Dynamics After Control has been Applied

For a saddle-node bifurcation, the center manifold dynamics of a one-dimensional system are given by equation VII.49, and have the form

$$\dot{z}_1 = F_\mu \mu_1 + Q_{cm}(\Pi_L) \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \end{bmatrix} + C_{cm}(\Pi_L, \Pi_Q) \begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1 z_1^2 \\ z_1^3 \end{bmatrix} + O^{(4+)} \tag{VII.95}$$

where $F_\mu \neq 0$, and where $F_z = 0$ is implicit in the equation. The question is, for systems of the form VII.95, when $F_\mu \neq 0$, does the system experience a saddle-node bifurcation at the origin? The answer is yes. Appendix D shows that a system of the form of equation VII.95 can be transformed by an appropriate coordinate transformation into a system of the form

$$\dot{\hat{z}}_1 = F_\mu \mu_1 + q^*_{cm_3} \hat{z}_1^2 + c^*_{cm_4} \hat{z}_1^3 + O^{(4+)} \tag{VII.96}$$

which is clearly of the form of equation VII.85. Appendix D also shows that the quadratic and cubic coefficients $q^*_{cm_3}$ and $c^*_{cm_4}$ can be written as functions of the elements of the coefficient matrices $Q_{cm}(\Pi_L)$ and $C_{cm}(\Pi_L, \Pi_Q)$, as

$$q^*_{cm_3} = q_{cm_3} \tag{VII.97}$$

$$c^*_{cm_4} = c_{cm_4} \tag{VII.98}$$

The coefficients q_{cm_3} and c_{cm_4} are taken from equation VII.60 and VII.70 respectively. So, if $F_\mu \neq 0$, and as long as $q_{cm_3} \neq 0$, the system exhibits a saddle-node bifurcation at the origin, which means that the system is subjected to quadratic instability.

The previous analysis assumed that $F_\mu \neq 0$ and $q_{cm_3} \neq 0$. Although F_μ is not subject to change, q_{cm_3} is a function of Π_L, which we can manipulate. If we could apply control and somehow eliminate the quadratic term by forcing $q_{cm_3} = 0$, would we be any better off? The stability of our equation would be governed by the cubic terms, and we would have

$$\dot{\hat{z}}_1 = F_\mu \mu_1 + c_{cm_4} \hat{z}_1^3 + O^{(4+)} \tag{VII.99}$$

Now the situation is very different. Regardless of the value of μ_1 our system only has one equilibrium point, at

$$\hat{z}^* = \sqrt[3]{\frac{-F_\mu \mu_1}{c_{cm_4}}} \tag{VII.100}$$

and the stability is determined purely by the sign of c_{cm_4}. (For $c_{cm_4} < 0$ the system is stable, while for $c_{cm_4} > 0$ the system is unstable.) So, our control strategy for a system which exhibits a saddle-node bifurcation will be to attempt to force $q_{cm_3} = 0$ with linear state feedback, and to force $c_{cm_4} = c^*_{cm_4} < 0$ with quadratic state feedback.

b. Determining the Linear Terms of the Center Manifold (Π_L)*, and the Linear Gains*

Now we want to determine how to manipulate Π_L so as to force q_{cm_3} to zero. Restating equation VII.60, we have

$$q_{cm_3} = q_{z_{P_3}} + q_{z_{m_2}} \Pi_{L_{1z_1}} + q_{z_{c_1}} \left(\Pi_{L_{1z_1}}\right)^2 \tag{VII.101}$$

which we will use to pick a value for Π_L. However, before we proceed to a theorem giving the solution, two important points need to be made. First, equation VII.101 only depends on $\Pi_{L_{1z_1}}$, leaving $\Pi_{L_{1\mu_1}}$ as an arbitrary free variable. Although Chapter VI showed that the first row of the matrix Π_L determined all the other rows of the matrix, as well as the linear gains, we do not need to use all the elements of the

first row in our attempt to force q_{cm_3} to zero in this case. Second, equation VII.101 may not have a solution for $q_{cm_3} = 0$. For example, the case of $q_{z_{P_3}} = 1$, $q_{z_{m_2}} = -1$ and $q_{z_{c_1}} = 2$ is illustrated in Figure 8, where no matter what value is picked for

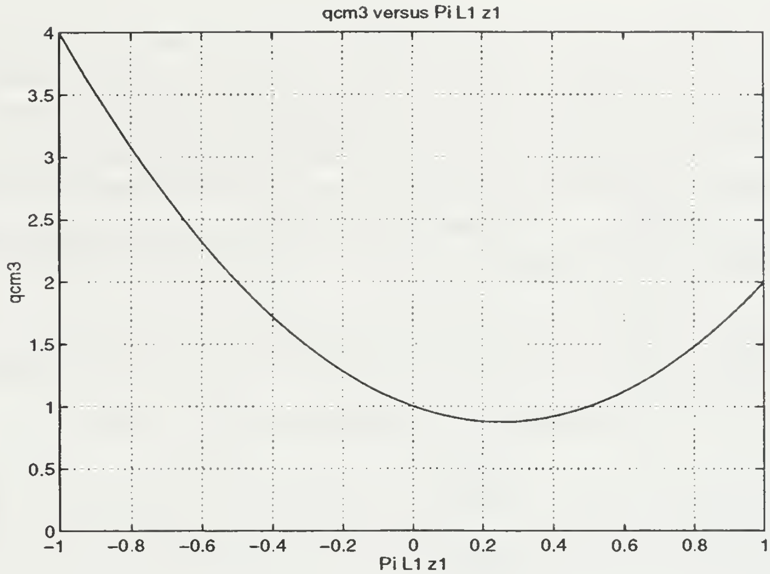

Figure 8. Uncancellable One-Dimensional Quadratic Terms

$\Pi_{L_{1z_1}}$, the coefficient q_{cm_3} cannot be forced to zero. If q_{cm_3} cannot be forced to zero by an appropriate choice of $\Pi_{L_{1z}}$, then the saddle-node bifurcation can not be eliminated with feedback, and hysteresis in the system is inevitable around the point of bifurcation. However, in this case, there may be some value in choosing $\Pi_{L_{1z_1}}$ such that the magnitude of q_{cm_3} is appropriately minimized, in an attempt to bound or minimize the hysteresis in the system. With those two points in mind, we come to a theorem.

Theorem C.4 (Saddle-Node Linear Gains) *For a control system in the quadratic normal form of equation VII.48, with $F_\mu \neq 0$, and with the vector of linear state feedback gains K_y chosen to stabilize the linearly controllable states y, the linear state feedback gain which forces the coefficient of the quadratic term z_1^2 to zero and suppresses the hysteresis associated with the accompanying saddle-node bifurcation is given by*

$$K_{z_1} = -K_{y_1}\Pi_{L_{1z_1}} \qquad \text{(VII.102)}$$

where the coefficient $\Pi_{L_{1z_1}}$ *is given by*

$$\Pi_{L_{1z_1}} = \frac{-q_{z_{m_2}} \pm \sqrt{\left(q_{z_{m_2}}\right)^2 - 4q_{z_{c_1}} q_{z_{P_3}}}}{2q_{z_{c_1}}} \tag{VII.103}$$

and where the coefficients $q_{z_{P_3}}$, $q_{z_{m_2}}$, *and* $q_{z_{c_1}}$ *are the appropriate elements of the coefficient matrices* $Q_{z_{P_1}}$, $Q_{z_{m_1}}$, *and* Q_{z_c} *from the one dimensional quadratic normal form equation VII.48. For those cases where equation VII.103 does not have a real solution, then there are no linear state feedback gains which can suppress the hysteresis associated with the saddle-node bifurcation. The remaining linear state feedback gains* K_{μ_1} *and* K_w *have no effect on the equivalent quadratic order dynamics and may be chosen arbitrarily, including being set to zero.*

Proof. From Chapter VI, in the Linear Center Manifold Solution theorem, for the one dimensional, co-dimension one case, we have

$$\begin{bmatrix} K_{\mu_1} & K_{z_1} \end{bmatrix} = \begin{bmatrix} \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ F_\mu & 0 \end{bmatrix}^p - \sum_{i=1}^{p} K_{y_i} \begin{bmatrix} 0 & 0 \\ F_\mu & 0 \end{bmatrix}^{i-1} \right) \tag{VII.104}$$

We also have

$$\begin{bmatrix} 0 & 0 \\ F_\mu & 0 \end{bmatrix}^j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{VII.105}$$

for $j \geq 2$. For $p = 1$, we have

$$\begin{bmatrix} K_{\mu_1} & K_{z_1} \end{bmatrix} = \begin{bmatrix} \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ F_\mu & 0 \end{bmatrix} - K_{y_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \tag{VII.106}$$

which, when multiplied out gives

$$K_{\mu_1} = F_\mu \Pi_{L_{1z_1}} - K_{y_1} \Pi_{L_{1\mu_1}} \tag{VII.107}$$

$$K_{z_1} = -K_{y_1} \Pi_{L_{1z_1}} \tag{VII.108}$$

For $p \geq 2$ we have

$$\begin{bmatrix} K_{\mu_1} & K_{z_1} \end{bmatrix} = \begin{bmatrix} \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} \end{bmatrix} \left(-K_{y_2} \begin{bmatrix} 0 & 0 \\ F_\mu & 0 \end{bmatrix} - K_{y_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \tag{VII.109}$$

which, when multiplied out gives

$$K_{\mu_1} = -F_\mu K_{y_2} \Pi_{L_{1z_1}} - K_{y_1} \Pi_{L_{1\mu_1}} \tag{VII.110}$$

$$K_{z_1} = -K_{y_1} \Pi_{L_{1z_1}} \tag{VII.111}$$

In both cases we recover equation VII.102, proving the first part of the theorem. Equation VII.103 is recovered by setting $q_{cm_3} = 0$ and solving equation VII.101 for $\Pi_{L_{1z_1}}$, proving the second part of the theorem. If no real solution exists, then the value of $\Pi_{L_{1z_1}}$ which minimizes the magnitude of q_{cm_3} is found by setting the derivative of equation VII.101 with respect to $\Pi_{L_{1z_1}}$ to zero, and solving for $\Pi_{L_{1z_1}}$, then plugging back in, which gives

$$q_{cm_{3_{\min}}} = -\frac{\left(q_{z_{m_2}}\right)^2 - 4q_{z_{c_1}} q_{z_{P_3}}}{4q_{z_{c_1}}} \tag{VII.112}$$

Now, there are two reasons why equation VII.103 can fail to have real solutions. If $q_{z_{c_1}} = 0$, there are no real solutions to equation VII.103, and by equation VII.112 we have $q_{cm_{3_{\min}}} \neq 0$, so the quadratic z_1^2 term and its associated hysteresis cannot be suppressed. Likewise, if $\left(q_{z_{m_2}}\right)^2 - 4q_{z_{c_1}} q_{z_{P_3}} < 0$, there are no real solutions to equation VII.103, and by equation VII.112 we have $q_{cm_{3_{\min}}} \neq 0$, so the quadratic z_1^2 term and its associated hysteresis cannot be suppressed in this case, either. This proves the third part of the theorem. Since equation VII.101 only depends on $\Pi_{L_{1z_1}}$, we have $\Pi_{L_{1\mu_1}}$ as a free variable, which can be chosen arbitrarily. Looking at equations VII.107 and VII.110, we see that K_{μ_1} follows $\Pi_{L_{1\mu_1}}$ and is arbitrary, and can always be set to zero if

$$\Pi_{L_{1\mu_1}} = \frac{F_\mu}{K_{y_1}} \Pi_{L_{1z_1}} \tag{VII.113}$$

for $p = 1$, or

$$\Pi_{L_{1\mu_1}} = -K_{y_2} \frac{F_\mu}{K_{y_1}} \Pi_{L_{1z_1}} \tag{VII.114}$$

for $p \geq 2$. Note that $K_{y_1} \neq 0$, since the vector of gains K_y is assumed to stabilize the linearly controllable states. (If $K_{y_1} = 0$, then the closed loop linearly controllable plant, given by $A + BK_y^T$ has a zero eigenvalue, and is not stable, violating our

assumption.) Finally, from Chapter VI, in the Center Manifold Theorem and related proof, we showed that K_w does not affect the calculation of Π_L since $\Omega_L = 0$, and is therefore arbitrary, and from the corollary to the Center Manifold Theorem, we have that K_w may be set to zero. This proves the last part of the theorem. ◃

c. *Determining the Quadratic Terms of the Center Manifold* (Π_Q), *and the Quadratic Gains*

Now that we have chosen Π_L, we want to determine how to manipulate Π_Q so as to force c_{cm_4} to the desired value $c^*_{cm_4}$. Restating equation VII.70, we have

$$c_{cm_4} = \tilde{c}_{z_4} + q_\beta \Pi_{Q_{1z_1^2}} \tag{VII.115}$$

with

$$q_\beta = q_{z_{m_2}} + 2q_{z_{c_1}} \Pi_{L_{1z_1}} \tag{VII.116}$$

The coefficient $\tilde{c}_{z_4}$ is determined by evaluating equation VII.65, which we rewrite here for convenience as

$$\begin{aligned} \tilde{C}_z(\Pi_L)\begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1 z_1^2 \\ z_1^3 \end{bmatrix} &= \left(Q_{z_{P_2}} N_3(\Omega_Q) + Q_{z_{m_2}} N_5(\Pi_L, \Omega_Q)\right)\begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1 z_1^2 \\ z_1^3 \end{bmatrix} \\ &+ f_z^{(3)}\left(\mu_1, z_1, w_{cm}^{(1)}, y_{cm}^{(1)}\right) + g_z^{(2)}\left(\mu_1, z_1, w_{cm}^{(1)}, y_{cm}^{(1)}\right) v^{(1)} + O^{(4+)} \end{aligned} \tag{VII.117}$$

which is

$$\tilde{C}_z(\Pi_L) = \begin{bmatrix} \tilde{c}_{z_1} & \tilde{c}_{z_2} & \tilde{c}_{z_3} & \tilde{c}_{z_4} \end{bmatrix} \tag{VII.118}$$

Knowing Π_L, the coefficient matrix $\tilde{C}_z(\Pi_L)$ is most easily evaluated for a specific system, following the method of Appendix C. (It is very difficult to evaluate $\tilde{C}_z(\Pi_L)$ for the general case, since the dimensions m and p of the vectors w and y respectively are not specified.) Now, if we desire the cubic coefficient of our dynamic system to have the value $c^*_{cm_4} < 0$, where we have chosen $c^*_{cm_4}$ to provide non-linear (cubic order) stability, then we can plug in and solve equation VII.115. However, before we proceed

to a theorem giving the solution, one important point needs to be made. Equation VII.115 only depends on $\Pi_{Q_{1z_1^2}}$, leaving $\Pi_{Q_{1\mu_1^2}}$ and $\Pi_{Q_{1\mu_1 z_1}}$ as arbitrary free variables. Although Chapter VI showed that the first row of the matrix Π_Q determined all the other rows of the matrix, which determined the quadratic gains, we do not need to use all the elements of the first row to force c_{cm_4} to our desired value $c^*_{cm_4}$ in this case. This brings us to the next theorem.

Theorem C.5 (Saddle Node Quadratic Gains) *For a control system in the quadratic normal form of equation VII.48, with $F_\mu \neq 0$, with the vector of linear state feedback gains K_y chosen to stabilize the linearly controllable states y, and with the linear state feedback gains K_w and quadratic state feedback gains $K_{\mu y(2)}$, $K_{zy(2)}$ and $K_{y(2)}$ set to zero, the quadratic state feedback gain which forces the coefficient of the cubic term z_1^3 to the desired value $c^*_{cm_4}$ is given by*

$$K_{z_1^2} = -K_{y_1}\Pi_{Q_{1z_1^2}} + \Gamma_{K_3} \tag{VII.119}$$

with the coefficient $\Pi_{Q_{1z_1^2}}$ given by

$$\Pi_{Q_{1z_1^2}} = \frac{c^*_{cm_4} - \tilde{c}_{z_4}}{q_{zm_2} + 2q_{zc_1}\Pi_{L_{1z_1}}} \tag{VII.120}$$

and with

$$\Gamma_{K_3} = -\Pi_{L_{1z_1}} q_{cm_3} \tag{VII.121}$$

for $p = 1$, and

$$\Gamma_{K_3} = K_{y_2}\Pi_{L_{1z_1}} q_{cm_3} \tag{VII.122}$$

for $p \geq 2$. The quadratic coefficient q_{cm_3} and the linear center manifold coefficient $\Pi_{L_{1z_1}}$ are determined by the choice of linear state feedback gains from the Saddle Node Linear Gains theorem, where q_{cm_3} is normally forced to zero. The quadratic coefficients q_{zm_2} and q_{zc_1} are the appropriate elements of the coefficient matrices Q_{zm_1}, and Q_{zc} from the one dimensional quadratic normal form equation VII.48, $c^*_{cm_4}$ is the desired value of the coefficient of the cubic term z_1^3 after stabilizing quadratic state feedback has been applied, and the cubic coefficient $\tilde{c}_{z_4}$ is calculated from the individual system being analyzed using equations VII.117 and VII.118. The remaining quadratic state feedback gains $K_{\mu_1^2}$ and $K_{\mu_1 z_1}$ have no effect on the equivalent cubic order dynamics and may be chosen arbitrarily, including being set to zero.

Proof. From the Quadratic Center Manifold Solution theorem in Chapter VI for the one dimensional, co-dimension one case, we have

$$\begin{bmatrix} K_{\mu_1^2} & K_{\mu_1 z_1} & K_{z_1^2} \end{bmatrix} = \begin{bmatrix} \Pi_{Q_{1\mu_1^2}} & \Pi_{Q_{1\mu_1 z_1}} & \Pi_{Q_{1z_1^2}} \end{bmatrix} D_K + \begin{bmatrix} \Gamma_{K_1} & \Gamma_{K_2} & \Gamma_{K_3} \end{bmatrix} \tag{VII.123}$$

with

$$D_K = D_\xi^p - \sum_{i=1}^{p} K_{y_i} D_\xi^{i-1} \tag{VII.124}$$

Appendix A gives

$$D_\xi = \begin{bmatrix} 0 & 0 & 0 \\ F_\mu & 0 & 0 \\ 0 & 2F_\mu & 0 \end{bmatrix} \tag{VII.125}$$

and we get

$$D_\xi^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2F_\mu^2 & 0 & 0 \end{bmatrix} \tag{VII.126}$$

and

$$D_\xi^j = 0 \tag{VII.127}$$

for $j \geq 3$, and where we have used the definition $D_\xi^0 = I$. Looking at the matrix D_K, we see that it is lower triangular with $-K_{y_1}$ on the main diagonal, regardless of the dimension of p. Since we are given that the gains K_y stabilize the linearly controllable states y, we have $K_{y_1} \neq 0$ by the arguments given in the proof of the Saddle Node Linear Gains theorem, and so the matrix D_K is invertible. Particularly, we have

$$K_{z_1^2} = -K_{y_1} \Pi_{Q_{1 z_1^2}} + \Gamma_{K_3} \tag{VII.128}$$

since D_K is lower triangular, which proves the first part of the theorem. Plugging $c_{cm_4} = c_{cm_4}^*$ into equation VII.115 and solving for $\Pi_{Q_{1 z_1^2}}$ proves the second part. Γ_{K_3} is calculated from equations VII.77, VII.83 and VII.84. Since we have

$$\Gamma_{K_3} = \Gamma_K^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{VII.129}$$

equations VII.83 and VII.84 become

$$\Gamma_{K_3} = -\Gamma_{z_1} (\Pi_L) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{VII.130}$$

for $p = 1$, and

$$\Gamma_{K_3} = K_{y_2}\Gamma_{z_1}\left(\Pi_L\right)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{VII.131}$$

for $p \geq 2$. Plugging in equation VII.77, we get

$$\Gamma_{z_1}\left(\Pi_L\right)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \Pi_{L_{1_{z1}}}\left(Q_{zP_1} + Q_{zm_1}M_1\left(\Pi_L\right) + Q_{zc}M_2\left(\Pi_L\right)\right)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{VII.132}$$

From Appendix C we have

$$M_1\left(\Pi_L\right) = \begin{bmatrix} \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} & 0 \\ 0 & \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} \end{bmatrix} \tag{VII.133}$$

and

$$M_2\left(\Pi_L\right) = \begin{bmatrix} \left(\Pi_{L_{1\mu_1}}\right)^2 & 2\left(\Pi_{L_{1\mu_1}}\right)\left(\Pi_{L_{1z_1}}\right) & \left(\Pi_{L_{1z_1}}\right)^2 \\ \left(F_\mu \Pi_{L_{1z_1}}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \tag{VII.134}$$

So, plugging in and evaluating gives

$$\Gamma_{K_3} = -\Pi_{L_{1_{z1}}}\left(q_{zP_3} + q_{zm_2}\Pi_{L_{1z_1}} + q_{zc_1}\left(\Pi_{L_{1z_1}}\right)^2\right) \tag{VII.135}$$

for $p = 1$, and

$$\Gamma_{K_3} = K_{y_2}\Pi_{L_{1_{z1}}}\left(q_{zP_3} + q_{zm_2}\Pi_{L_{1z_1}} + q_{zc_1}\left(\Pi_{L_{1z_1}}\right)^2\right) \tag{VII.136}$$

for $p \geq 2$. But, equation VII.101 defines the common term in parentheses as q_{cm_3}, which proves the third part of the theorem. Since equation VII.115 only depends on $\Pi_{Q_{1z_1^2}}$, we have $\Pi_{Q_{1\mu_1^2}}$ and $\Pi_{Q_{1\mu_1 z_1}}$ as free variables, which can be chosen arbitrarily. Since D_K is invertible, we can always choose $K_{\mu_1^2}$ and $K_{\mu_1 z_1}$ arbitrarily, including

$K_{\mu_1^2} = 0$ and $K_{\mu_1 z_1} = 0$, which results in values of $\Pi_{Q_{1\mu_1^2}}$ and $\Pi_{Q_{1\mu_1 z_1}}$ determined by the equation

$$\begin{bmatrix} \Pi_{Q_{1\mu_1^2}} & \Pi_{Q_{1\mu_1 z_1}} & \Pi_{Q_{1z_1^2}} \end{bmatrix} = \left(\begin{bmatrix} K_{\mu_1^2} & K_{\mu_1 z_1} & K_{z_1^2} \end{bmatrix} - \begin{bmatrix} \Gamma_{K_1} & \Gamma_{K_2} & \Gamma_{K_3} \end{bmatrix} \right) D_K^{-1} \tag{VII.137}$$

which proves the last part of the theorem. ◁

4. One Dimensional Degenerate Bifurcations

The general dynamics on a one dimensional center manifold were given by equation VII.49, which we restate here for convenience,

$$\dot{z}_1 = F_\mu \mu_1 + Q_{cm}(\Pi_L) \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \end{bmatrix} + C_{cm}(\Pi_L, \Pi_Q) \begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1 z_1^2 \\ z_1^3 \end{bmatrix} + O^{(4+)} \tag{VII.138}$$

with $\mu = [\mu_1] \in R^1$ and $z = [z_1] \in R^1$. The case of $F_\mu \neq 0$ yielded a saddle-node bifurcation, which was investigated in the previous section. When $F_\mu = 0$ we have a degenerate case, which yields different types of one dimensional bifurcations and other degenerate conditions, depending on the values of the quadratic and cubic coefficients. Since there are many different cases depending on the value of the coefficients, we will not attempt a comprehensive treatment here. Instead, we will attempt a brief overview of the different possibilities, followed by specific investigations of certain special cases of interest.

When the quadratic terms in equation VII.138 are dominant, that is, when $F_\mu = 0$ and the quadratic coefficients are non-zero, the dynamics on the center manifold in the local vicinity of the origin are characterized by equations of the form

$$\dot{z} = Q_\mu \mu^2 + Q_{\mu z} \mu z + Q_z z^2 + O^{(3+)} \tag{VII.139}$$

where $\mu \in R^1$, $z \in R^1$ and Q_μ, $Q_{\mu z}$ and Q_z are scalar constants. Solving for the local equilibrium points of equation VII.139 by finding z^*, such that $\dot{z} = 0$ when $z = z^*$,

yields two possible equilibrium points

$$z_+^* = \mu\left(\frac{-Q_{\mu z} + \sqrt{(Q_{\mu z})^2 - 4Q_zQ_\mu}}{2Q_z}\right) \qquad \text{(VII.140)}$$

and

$$z_-^* = \mu\left(\frac{-Q_{\mu z} - \sqrt{(Q_{\mu z})^2 - 4Q_zQ_\mu}}{2Q_z}\right) \qquad \text{(VII.141)}$$

However, these two local equilibrium points do not always exist. We list the possible cases as:

- For $Q_z \neq 0$ and $(Q_{\mu z})^2 - 4Q_zQ_\mu > 0$, two distinct local equilibrium points exist (except at $\mu = 0$, where there is only one). This situation characterizes a transcritical bifurcation.
- For $(Q_{\mu z})^2 - 4Q_zQ_\mu < 0$, only one equilibrium point exists at $\mu = 0$, and no equilibrium points exist for $\mu \neq 0$. This situation characterizes the case of an isolated equilibrium point.
- For $Q_z = 0$, our solution method breaks down, and we revert to solving equation VII.139 directly. We find that a single local equilibrium point exists at $z^* = -\frac{Q_\mu}{Q_{\mu z}}\mu$ (for $Q_{\mu z} \neq 0$), except when $\mu = 0$, in which case $z^* = arbitrary$. This situation characterizes a pitchfork bifurcation.

Even more degenerate cases are possible; however, we will not consider them here. Instead we will consider the general control strategy for degenerate cases of this type, which is to attempt to apply state feedback to force $Q_z = 0$ and to force $Q_{\mu z}$ to a non-zero value (hopefully a desired non-zero value) so as to eliminate the inherent instability associated with the z^2 term in one-dimensional dynamics.

For any of the degenerate one dimensional cases above with $F_\mu = 0$, we can write equation VII.138 as

$$\begin{aligned}
\dot{z}_1 &= Q_{cm}(\Pi_L)\begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \end{bmatrix} + C_{cm}(\Pi_L, \Pi_Q)\begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1 z_1^2 \\ z_1^3 \end{bmatrix} + O^{(4+)} \qquad \text{(VII.142)} \\
&= q_{cm_1}\mu_1^2 + q_{cm_2}\mu_1 z_1 + q_{cm_3} z_1^2 \\
&+ c_{cm_1}\mu_1^3 + c_{cm_2}\mu_1^2 z_1 + c_{cm_1}\mu_1 z_1^2 + c_{cm_1} z_1^3 + O^{(4+)}
\end{aligned}$$

Appendix D shows that a system of the form of equation VII.142 can be transformed by an appropriate coordinate transformation into more than one normal form. However, all the possible normal forms have the property that the quadratic coefficients q_{cm_1}, q_{cm_2}, and q_{cm_3} remain unaltered. We choose as our normal form a system of the form

$$\begin{aligned} \dot{\hat{z}}_1 &= q_{cm_1}\mu_1^2 + q_{cm_2}\mu_1\hat{z}_1 + q_{cm_3}\hat{z}_1^2 \\ &+ \left(c_{cm_2} - 2\left(\frac{q_{cm_3}}{q_{cm_2}}\right)c_{cm_1} - 2\left(\frac{q_{cm_1}}{q_{cm_2}}\right)c_{cm_3}\right)\mu_1^2\hat{z}_1 + c_{cm_4}\hat{z}_1^3 + O^{(4+)} \end{aligned} \tag{VII.143}$$

Now we can look at what the effects of applying control might be.

a. Desired Dynamics After Control has been Applied

Previous sections showed that q_{cm_1}, q_{cm_2} and q_{cm_3} are functions of Π_L, which we can manipulate. If we could apply control and eliminate the $\hat{z}_1^2$ term by forcing q_{cm_3} to zero, eliminate the μ_1^2 term by forcing q_{cm_1} to zero, and ensure that the mixed term $\mu_1\hat{z}_1$ existed by forcing q_{cm_2} to be non-zero (and preferably a desired value), would we be any better off? The dynamics of our system would be

$$\dot{\hat{z}}_1 = q_{cm_2}\mu_1\hat{z}_1 + c_{cm_2}\mu_1^2\hat{z}_1 + c_{cm_4}\hat{z}_1^3 + O^{(4+)}. \tag{VII.144}$$

which exhibits a pitchfork bifurcation. The side of the origin that the bifurcation occurs on is determined by the sign of q_{cm_2}, and the criticality (subcritical or supercritical) is determined by the sign of c_{cm_4}, while c_{cm_2} is the coefficient of a higher order term and does not influence the local dynamics in the vicinity of the origin. So, our control strategy for the case of a degenerate one dimensional bifurcation will be to try and cancel the appropriate quadratic terms with linear state feedback, and to produce a negative cubic term with quadratic state feedback, so as to have the closed loop system undergo a supercritical pitchfork bifurcation.

b. Determining the Linear Terms of the Center Manifold (Π_L), and the Linear Gains

Now we want to determine how to manipulate Π_L so as to force q_{cm_1} and q_{cm_3} to zero if possible, and to ensure that q_{cm_2} is non-zero. Restating equations

VII.58, VII.59 and VII.60 for the case where $F_\mu = 0$, we have

$$q_{cm_1} = q_{zP_1} + q_{zm_1}\Pi_{L_{1\mu_1}} + q_{zc_1}\left(\Pi_{L_{1\mu_1}}\right)^2 \tag{VII.145}$$

$$q_{cm_2} = q_{zP_2} + q_{zm_2}\Pi_{L_{1\mu_1}} + q_{zm_1}\Pi_{L_{1z_1}} + 2q_{zc_1}\Pi_{L_{1\mu_1}}\Pi_{L_{1z_1}} \tag{VII.146}$$

$$q_{cm_3} = q_{zP_3} + q_{zm_2}\Pi_{L_{1z_1}} + q_{zc_1}\left(\Pi_{L_{1z_1}}\right)^2 \tag{VII.147}$$

which we will use to pick a value for Π_L. However, before we proceed to a theorem giving the solution, two important points need to be made. First, equations VII.145, VII.146 and VII.147 are three equations in two unknowns, $\Pi_{L_{1\mu_1}}$ and $\Pi_{L_{1z_1}}$, so in general we will not be able to force all three coefficients q_{cm_1}, q_{cm_2} and q_{cm_3} to desired values. Second, equations VII.145 and VII.147 may not have a solution for $q_{cm_1} = 0$ and $q_{cm_3} = 0$. This is similar to the situation illustrated for the saddle-node bifurcation in Figure 8. If q_{cm_3} cannot be forced to zero by an appropriate choice of $\Pi_{L_{1z_1}}$, then the inherent instability of the quadratic term z_1^2 in the system can not be eliminated with feedback, and hysteresis is inevitable around the point of bifurcation. However, in this case, there may be some value in choosing $\Pi_{L_{1z_1}}$ such that the magnitude of q_{cm_3} is appropriately minimized, in an attempt to bound or minimize the hysteresis in the system. (Also, note that if q_{cm_1} cannot be forced to zero by an appropriate choice of $\Pi_{L_{1\mu_1}}$, then further coordinate changes are needed to analyze the system before cubic stability can be imposed.) With those two points in mind, we come to the next theorem.

Theorem C.6 (1D Degenerate Linear Gains) *For a control system in the quadratic normal form of equation VII.48, with $F_\mu = 0$, and with the vector of linear state feedback gains K_y chosen to stabilize the linearly controllable states y, the linear state feedback gains which force the coefficients of the quadratic terms μ_1^2 and z_1^2 to zero and suppress the inherent quadratic order instability in a one dimensional system are given by*

$$K_{\mu_1} = -K_{y_1}\Pi_{L_{1\mu_1}} \tag{VII.148}$$

and

$$K_{z_1} = -K_{y_1}\Pi_{L_{1z_1}} \tag{VII.149}$$

The coefficients $\Pi_{L_{1\mu_1}}$ *and* $\Pi_{L_{1z_1}}$ *are given by*

$$\Pi_{L_{1\mu_1}} = \frac{-q_{z_{m_1}} \pm \sqrt{\left(q_{z_{m_1}}\right)^2 - 4q_{z_{c_1}} q_{z_{P_1}}}}{2q_{z_{c_1}}} \tag{VII.150}$$

and

$$\Pi_{L_{1z_1}} = \frac{-q_{z_{m_2}} \pm \sqrt{\left(q_{z_{m_2}}\right)^2 - 4q_{z_{c_1}} q_{z_{P_3}}}}{2q_{z_{c_1}}} \tag{VII.151}$$

where the coefficients $q_{z_{P_1}}$, $q_{z_{m_1}}$, $q_{z_{P_3}}$, $q_{z_{m_2}}$, *and* $q_{z_{c_1}}$ *are the appropriate elements of the coefficient matrices* $Q_{z_{P_1}}$, $Q_{z_{m_1}}$, *and* Q_{z_c} *from the one dimensional quadratic normal form equation VII.48. For those cases where equation VII.151 does not have a real solution, then there are no linear state feedback gains which can eliminate the quadratic order instability associated with the* z_1^2 *term.*

Proof. From Chapter VI, in the Linear Center Manifold Solution theorem, for the one dimensional, co-dimension one case, we have

$$\begin{bmatrix} K_{\mu_1} & K_{z_1} \end{bmatrix} = \begin{bmatrix} \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ F_\mu & 0 \end{bmatrix}^p - \sum_{i=1}^{p} K_{y_i} \begin{bmatrix} 0 & 0 \\ F_\mu & 0 \end{bmatrix}^{i-1} \right) \tag{VII.152}$$

Since $F_\mu = 0$, we have

$$\begin{bmatrix} K_{\mu_1} & K_{z_1} \end{bmatrix} = \begin{bmatrix} \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} \end{bmatrix} \left(-K_{y_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \tag{VII.153}$$

where we have used the definition $\begin{bmatrix} 0 & 0 \\ F_\mu & 0 \end{bmatrix}^0 = I$. This proves equations VII.148 and VII.149. Equation VII.150 is recovered by setting $q_{cm_1} = 0$ and solving equation VII.145 for $\Pi_{L_{1\mu_1}}$, and equation VII.151 is recovered by setting $q_{cm_3} = 0$ and solving equation VII.147 for $\Pi_{L_{1z_1}}$. If no real solution exists, then the value of $\Pi_{L_{1z_1}}$ which minimizes the magnitude of q_{cm_3} is found by setting the derivative of equation VII.147 with respect to $\Pi_{L_{1z_1}}$ to zero, and solving for $\Pi_{L_{1z_1}}$, then plugging back in, which gives

$$q_{cm_{3_{\min}}} = -\frac{\left(q_{z_{m_2}}\right)^2 - 4q_{z_{c_1}} q_{z_{P_3}}}{4q_{z_{c_1}}} \tag{VII.154}$$

Now, there are two reasons why equation VII.151 can fail to have real solutions. If $q_{z_{c_1}} = 0$, there are no real solutions to equation VII.151, and by equation VII.154 we have $q_{cm_{3_{\mathrm{min}}}} \neq 0$, so the quadratic z_1^2 term and its associated hysteresis cannot be suppressed. Likewise, if $\left(q_{z_{m_2}}\right)^2 - 4q_{z_{c_1}} q_{z_{P_3}} < 0$, there are no real solutions to equation VII.151, and by equation VII.154 we have $q_{cm_{3_{\mathrm{min}}}} \neq 0$, so the quadratic z_1^2 term and its associated hysteresis cannot be suppressed in this case, either. This completes the proof of the theorem. ◁

Now we state a lemma which resolves the ambiguity in whether to choose the plus or minus sign in the solutions for $\Pi_{L_{1_{\mu_1}}}$ and $\Pi_{L_{1_{z_1}}}$ given in the above theorem.

Lemma C.7 (Resolving Ambiguity in Π_L) *The ambiguity in the choice of values for $\Pi_{L_{1_{\mu_1}}}$ and $\Pi_{L_{1_{z_1}}}$ can be resolved by picking the most suitable value of q_{cm_2} from the four possible choices. The value of q_{cm_2} is given by equation VII.146 as*

$$q_{cm_2} = q_{z_{P_2}} + q_{z_{m_2}} \Pi_{L_{1_{\mu_1}}} + q_{z_{m_1}} \Pi_{L_{1_{z_1}}} + 2q_{z_{c_1}} \Pi_{L_{1_{\mu_1}}} \Pi_{L_{1_{z_1}}} \tag{VII.155}$$

and the four possible cases are:
Case 1:

$$\Pi_{L_{1_{\mu_1}}} = \Pi_{L_{1_{\mu+}}} \tag{VII.156}$$
$$\Pi_{L_{1_{z_1}}} = \Pi_{L_{1_{z+}}} \tag{VII.157}$$

Case 2:

$$\Pi_{L_{1_{\mu_1}}} = \Pi_{L_{1_{\mu+}}} \tag{VII.158}$$
$$\Pi_{L_{1_{z_1}}} = \Pi_{L_{1_{z-}}} \tag{VII.159}$$

Case 3:

$$\Pi_{L_{1_{\mu_1}}} = \Pi_{L_{1_{\mu-}}} \tag{VII.160}$$
$$\Pi_{L_{1_{z_1}}} = \Pi_{L_{1_{z+}}} \tag{VII.161}$$

Case 4:

$$\Pi_{L_{1_{\mu_1}}} = \Pi_{L_{1_{\mu-}}} \tag{VII.162}$$
$$\Pi_{L_{1_{z_1}}} = \Pi_{L_{1_{z-}}} \tag{VII.163}$$

where the notation indicates which choice is made, as follows

$$\Pi_{L_{1\mu+}} = \frac{-q_{z_{m_1}} + \sqrt{\left(q_{z_{m_1}}\right)^2 - 4q_{z_{c_1}}q_{z_{P_1}}}}{2q_{z_{c_1}}} \tag{VII.164}$$

$$\Pi_{L_{1\mu-}} = \frac{-q_{z_{m_1}} - \sqrt{\left(q_{z_{m_1}}\right)^2 - 4q_{z_{c_1}}q_{z_{P_1}}}}{2q_{z_{c_1}}} \tag{VII.165}$$

$$\Pi_{L_{1z+}} = \frac{-q_{z_{m_2}} + \sqrt{\left(q_{z_{m_2}}\right)^2 - 4q_{z_{c_1}}q_{z_{P_3}}}}{2q_{z_{c_1}}} \tag{VII.166}$$

$$\Pi_{L_{1z-}} = \frac{-q_{z_{m_2}} - \sqrt{\left(q_{z_{m_2}}\right)^2 - 4q_{z_{c_1}}q_{z_{P_3}}}}{2q_{z_{c_1}}} \tag{VII.167}$$

Proof. Since there are three equations in two unknowns, one equation will not be able to be solved exactly in general. It is desired to exactly fix $q_{cm_1} = 0$ and $q_{cm_3} = 0$, but it is only required to have q_{cm_2} have the correct sign to put the bifurcation on the desired side of the origin. So, there are four possible cases which achieve $q_{cm_1} = 0$ and $q_{cm_3} = 0$, and we choose the case which produces the desired sign for q_{cm_2} (if it is one or more of the possible choices), which has the magnitude closest to our desired value. ◁

c. *Determining the Quadratic Terms of the Center Manifold* (Π_Q), *and the Quadratic Gains*

Now that we have chosen Π_L, we want to determine how to manipulate Π_Q so as to force c_{cm_4} to the desired value $c^*_{cm_4}$. Looking at equation VII.70 for c_{cm_4}, we have

$$c_{cm_4} = \tilde{c}_{z_4} + q_\beta \Pi_{Q_{1z_1^2}} \tag{VII.168}$$

with

$$q_\beta = q_{z_{m_2}} + 2q_{z_{c_1}}\Pi_{L_{1z_1}} \tag{VII.169}$$

The coefficient $\tilde{c}_{z_4}$ is determined by evaluating equation VII.65, which we rewrite for convenience as

$$\tilde{C}_z(\Pi_L)\begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1 z_1^2 \\ z_1^3 \end{bmatrix} = \left(Q_{zP_2} N_3(\Omega_Q) + Q_{zm_2} N_5(\Pi_L, \Omega_Q)\right)\begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1 z_1^2 \\ z_1^3 \end{bmatrix} + f_z^{(3)}\left(\mu_1, z_1, w_{cm}^{(1)}, y_{cm}^{(1)}\right) + g_z^{(2)}\left(\mu_1, z_1, w_{cm}^{(1)}, y_{cm}^{(1)}\right) v^{(1)} + O^{(4+)} \qquad \text{(VII.170)}$$

with

$$\tilde{C}_z(\Pi_L) = \begin{bmatrix} \tilde{c}_{z_1} & \tilde{c}_{z_2} & \tilde{c}_{z_3} & \tilde{c}_{z_4} \end{bmatrix} \qquad \text{(VII.171)}$$

Knowing Π_L, the coefficient matrix $\tilde{C}_z(\Pi_L)$ is most easily evaluated for a specific system, following the method of Appendix C. (It is very difficult to evaluate $\tilde{C}_z(\Pi_L)$ for the general case, since the dimensions m and p of the vectors w and y respectively are not specified.) Now, if we desire the cubic coefficient of our dynamic system to have the value $c_{cm_4}^* < 0$, where we have chosen $c_{cm_4}^*$ to provide non-linear (cubic order) stability, then we can plug in and solve equation VII.168. However, before we proceed to a theorem giving the solution, one important point needs to be made. Equation VII.168 only depends on $\Pi_{Q_{1z_1^2}}$, leaving $\Pi_{Q_{1\mu_1^2}}$ and $\Pi_{Q_{1\mu_1 z_1}}$ as arbitrary free variables. Although Chapter VI showed that the first row of the matrix Π_Q determined all the other rows of the matrix, which determined the quadratic gains, we do not need to use all the elements of the first row to force c_{cm_4} to our desired value $c_{cm_4}^*$ in this case. This brings us to the next theorem.

Theorem C.8 (Degenerate 1D Quadratic Gains) *For a control system in the quadratic normal form of equation VII.48, with $F_\mu = 0$, with the vector of linear state feedback gains K_y chosen to stabilize the linearly controllable states y, and with the linear state feedback gains K_w and quadratic state feedback gains $K_{\mu y^{(2)}}$, $K_{zy^{(2)}}$ and $K_{y^{(2)}}$ set to zero, the quadratic state feedback gain which forces the coefficient of the cubic term z_1^3 to the desired value $c_{cm_4}^*$ is given by*

$$K_{z_1^2} = -K_{y_1}\Pi_{Q_{1z_1^2}} + \Gamma_{K_3} \qquad \text{(VII.172)}$$

with the coefficient $\Pi_{Q_{1z_1^2}}$ *given by*

$$\Pi_{Q_{1z_1^2}} = \frac{c^*_{cm_4} - \tilde{c}_{z_4}}{q_{z_{m_2}} + 2q_{z_{c_1}}\Pi_{L_{1z_1}}} \tag{VII.173}$$

and where

$$\Gamma_{K_3} = -\Pi_{L_{1z_1}} q_{cm_3} \tag{VII.174}$$

for $p = 1$, *and*

$$\Gamma_{K_3} = K_{y_2}\Pi_{L_{1z_1}} q_{cm_3} \tag{VII.175}$$

for $p \geq 2$. *The quadratic coefficient* q_{cm_3} *and the linear center manifold coefficient* $\Pi_{L_{1z_1}}$ *are determined by the choice of linear state feedback gains from the Degenerate 1D Linear Gains theorem, where* q_{cm_3} *is normally forced to zero. The quadratic coefficients* $q_{z_{m_2}}$ *and* $q_{z_{c_1}}$ *are the appropriate elements of the coefficient matrices* $Q_{z_{m_1}}$, *and* Q_{z_c} *from the one dimensional quadratic normal form equation VII.48,* $c^*_{cm_4}$ *is the desired value of the coefficient of the cubic term* z_1^3 *after stabilizing quadratic state feedback has been applied, and the cubic coefficient* $\tilde{c}_{z_4}$ *is calculated from the individual system being analyzed using equations VII.170 and VII.171. The remaining quadratic state feedback gains* $K_{\mu_1^2}$ *and* $K_{\mu_1 z_1}$ *have no effect on the equivalent cubic order dynamics and may be chosen arbitrarily, including being set to zero.*

Proof. From the Quadratic Center Manifold Solution theorem in Chapter VI for the one dimensional, co-dimension one case, we have

$$\begin{bmatrix} K_{\mu_1^2} & K_{\mu_1 z_1} & K_{z_1^2} \end{bmatrix} = \begin{bmatrix} \Pi_{Q_{1\mu_1^2}} & \Pi_{Q_{1\mu_1 z_1}} & \Pi_{Q_{1z_1^2}} \end{bmatrix} D_K + \begin{bmatrix} \Gamma_{K_1} & \Gamma_{K_2} & \Gamma_{K_3} \end{bmatrix} \tag{VII.176}$$

with

$$D_K = D_\xi^p - \sum_{i=1}^{p} K_{y_i} D_\xi^{i-1} \tag{VII.177}$$

Appendix A gives

$$D_\xi = \begin{bmatrix} 0 & 0 & 0 \\ F_\mu & 0 & 0 \\ 0 & 2F_\mu & 0 \end{bmatrix} \tag{VII.178}$$

and since $F_\mu = 0$ we get

$$D_K = -K_{y_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{VII.179}$$

where we have used the definition $D_\xi^0 = I$. This gives

$$\begin{bmatrix} K_{\mu_1^2} & K_{\mu_1 z_1} & K_{z_1^2} \end{bmatrix} \tag{VII.180}$$

$$= \begin{bmatrix} \Pi_{Q_{1\mu_1^2}} & \Pi_{Q_{1\mu_1 z_1}} & \Pi_{Q_{1z_1^2}} \end{bmatrix} \left(-K_{y_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$+ \begin{bmatrix} \Gamma_{K_1} & \Gamma_{K_2} & \Gamma_{K_3} \end{bmatrix}$$

which proves equation VII.172. Plugging $c_{cm_4} = c^*_{cm_4}$ into equation VII.168 and solving for $\Pi_{Q_{1z_1^2}}$ proves the second part. Γ_{K_3} is calculated from equations VII.77, VII.83 and VII.84. Since we have

$$D_\xi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{VII.181}$$

and the definition $D_\xi^0 = I$, equations VII.83 and VII.84 become

$$\Gamma_K^T = -\Gamma_{z_1}(\Pi_L) \tag{VII.182}$$

for $p = 1$, and

$$\Gamma_K^T = K_{y_2}\Gamma_{z_1}(\Pi_L) \tag{VII.183}$$

for $p \geq 2$. Plugging in equation VII.77, we get

$$\Gamma_{z_1}(\Pi_L) = \Pi_{L_{1_{z1}}} \left(Q_{z_{P_1}} + Q_{z_{m_1}} M_1(\Pi_L) + Q_{z_c} M_2(\Pi_L) \right) \tag{VII.184}$$

From Appendix C we have

$$M_1(\Pi_L) = \begin{bmatrix} \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} & 0 \\ 0 & \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} \end{bmatrix} \tag{VII.185}$$

and

$$M_2(\Pi_L) = \begin{bmatrix} \left(\Pi_{L_{1\mu_1}}\right)^2 & 2\left(\Pi_{L_{1\mu_1}}\right)\left(\Pi_{L_{1z_1}}\right) & \left(\Pi_{L_{1z_1}}\right)^2 \\ \left(F_\mu \Pi_{L_{1z_1}}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(VII.186)}$$

So, plugging in and evaluating with $F_\mu = 0$ gives

$$\Gamma_{K_1} = -\Pi_{L_{1_{z1}}}\left(q_{zP_1} + q_{zm_1}\Pi_{L_{1\mu_1}} + q_{zc_1}\left(\Pi_{L_{1\mu_1}}\right)^2\right) \quad \text{(VII.187)}$$

$$\Gamma_{K_2} = -\Pi_{L_{1_{z1}}}\left(q_{zP_2} + q_{zm_2}\Pi_{L_{1\mu_1}} + q_{zm_1}\Pi_{L_{1z_1}} + 2q_{zc_1}\Pi_{L_{1\mu_1}}\Pi_{L_{1z_1}}\right) \quad \text{(VII.188)}$$

$$\Gamma_{K_3} = -\Pi_{L_{1_{z1}}}\left(q_{zP_3} + q_{zm_2}\Pi_{L_{1z_1}} + q_{zc_1}\left(\Pi_{L_{1z_1}}\right)^2\right) \quad \text{(VII.189)}$$

for $p = 1$, and

$$\Gamma_{K_1} = K_{y_2}\Pi_{L_{1_{z1}}}\left(q_{zP_1} + q_{zm_1}\Pi_{L_{1\mu_1}} + q_{zc_1}\left(\Pi_{L_{1\mu_1}}\right)^2\right) \quad \text{(VII.190)}$$

$$\Gamma_{K_2} = K_{y_2}\Pi_{L_{1_{z1}}}\left(q_{zP_2} + q_{zm_2}\Pi_{L_{1\mu_1}} + q_{zm_1}\Pi_{L_{1z_1}} + 2q_{zc_1}\Pi_{L_{1\mu_1}}\Pi_{L_{1z_1}}\right) \quad \text{(VII.191)}$$

$$\Gamma_{K_3} = K_{y_2}\Pi_{L_{1_{z1}}}\left(q_{zP_3} + q_{zm_2}\Pi_{L_{1z_1}} + q_{zc_1}\left(\Pi_{L_{1z_1}}\right)^2\right) \quad \text{(VII.192)}$$

for $p \geq 2$. But, equation VII.147 defines

$$q_{cm_3} = \left(q_{zP_3} + q_{zm_2}\Pi_{L_{1z_1}} + q_{zc_1}\left(\Pi_{L_{1z_1}}\right)^2\right)$$

which proves the next part of the theorem. Since equation VII.168 only depends on $\Pi_{Q_{1z_1^2}}$, we have $\Pi_{Q_{1\mu_1^2}}$ and $\Pi_{Q_{1\mu_1 z_1}}$ as free variables, which can be chosen arbitrarily. Since D_K is invertible, we can always choose $K_{\mu_1^2}$ and $K_{\mu_1 z_1}$ arbitrarily, including $K_{\mu_1^2} = 0$ and $K_{\mu_1 z_1} = 0$, which results in values of $\Pi_{Q_{1\mu_1^2}}$ and $\Pi_{Q_{1\mu_1 z_1}}$ determined by the equation

$$\begin{bmatrix} \Pi_{Q_{1\mu_1^2}} & \Pi_{Q_{1\mu_1 z_1}} & \Pi_{Q_{1z_1^2}} \end{bmatrix} = \left(\begin{bmatrix} K_{\mu_1^2} & K_{\mu_1 z_1} & K_{z_1^2} \end{bmatrix} - \begin{bmatrix} \Gamma_{K_1} & \Gamma_{K_2} & \Gamma_{K_3} \end{bmatrix}\right) D_K^{-1} \quad \text{(VII.193)}$$

which proves the last part of the theorem. ◁

5. Controlling Two Dimensional Bifurcations

In this section, we will look at the stabilization of various bifurcations occurring in two dimensions, which have one parameter (again, "co-dimension one" bifurcations). These bifurcations include the Hopf bifurcation, the double zero bifurcation, and the two-zeros bifurcation. Because they are two dimensional, these bifurcations are less amenable than one dimensional bifurcations to being analyzed as a family. Nonetheless, some analysis can still be done on them as a group.

For a system exhibiting a linearly uncontrollable, two dimensional, co-dimension one bifurcation, the quadratic normal form of the z dynamics of the control system is

$$\begin{aligned}
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= F_\mu \mu_1 + F_z \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad \text{(VII.194)} \\
&+ Q_{zP_1} \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ \mu_1 z_2 \\ z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix} + Q_{zm_1} \begin{bmatrix} \mu_1 y_1 \\ z_1 y_1 \\ z_2 y_1 \end{bmatrix} + Q_{zc} \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \\
&+ Q_{zP_2} \begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix} + Q_{zP_2} \begin{bmatrix} w^{(2)} \end{bmatrix} + Q_{zm_2} \begin{bmatrix} w y_1 \end{bmatrix} \\
&+ f_z^{(3)}(\mu_1, z_1, z_2, w, y) + g_z^{(2)}(\mu_1, z_1, z_2, w, y)\, v + O^{(4+)}
\end{aligned}$$

and the dynamics on the center manifold are given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = F_\mu \mu_1 + F_z \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + Q_{cm}(\Pi_L) \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ \mu_1 z_2 \\ z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix} + C_{cm}(\Pi_L, \Pi_Q) \begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1^2 z_2 \\ \mu_1 z_1^2 \\ \mu_1 z_1 z_2 \\ \mu_1 z_2^2 \\ z_1^3 \\ z_1^2 z_2 \\ z_1 z_2^2 \\ z_2^3 \end{bmatrix} + O^{(4+)} \tag{VII.195}$$

where we have used $\mu = [\mu_1] \in R^1$, $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in R^2$, $w \in R^m$ and $y \in R^p$. The rows of the matrix $F_\mu \in R^{2\times 1}$ depend on the rows of the matrix $F_z \in R^{2\times 2}$, that is any non-zero row in F_z forces the corresponding row in F_μ to be zero. The eigenvalues of the matrix F_z are both required to have zero real parts, and there are three possible ways this can happen. For

$$F_z = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \tag{VII.196}$$

where $\omega_0 \neq 0$, then

$$F_\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{VII.197}$$

and together these two matrices characterize a Hopf bifurcation. For

$$F_z = \begin{bmatrix} 0 & \lambda_0 \\ 0 & 0 \end{bmatrix} \tag{VII.198}$$

where $\lambda_0 \neq 0$, then

$$F_\mu = \begin{bmatrix} 0 \\ \beta \end{bmatrix} \tag{VII.199}$$

and together these two matrices characterize a co-dimension one double-zero bifurcation. Finally, for

$$F_z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{VII.200}$$

then

$$F_\mu = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{VII.201}$$

and together these two matrices characterize a co-dimension one two zeroes bifurcation. It is worth noting that Hopf bifurcations are generic for the co-dimension one case, that is, one parameter is all that is required to completely characterize their dynamics. However, both double-zero bifurcations and two zeroes bifurcations require more than one parameter to fully characterize their dynamics, and so these co-dimension one cases can be considered degenerate forms.

The coefficient matrices Q_{zP_1}, Q_{zm_1} and Q_{zc} in equation VII.194 are given by

$$Q_{zP_1} = \begin{bmatrix} q_{zP_{11}} & q_{zP_{12}} & q_{zP_{13}} & q_{zP_{14}} & q_{zP_{15}} & q_{zP_{16}} \\ q_{zP_{21}} & q_{zP_{22}} & q_{zP_{23}} & q_{zP_{24}} & q_{zP_{25}} & q_{zP_{26}} \end{bmatrix} \tag{VII.202}$$

$$Q_{zm_1} = \begin{bmatrix} q_{zm_{11}} & q_{zm_{12}} & q_{zm_{13}} \\ q_{zm_{21}} & q_{zm_{22}} & q_{zm_{23}} \end{bmatrix} \tag{VII.203}$$

$$Q_{zc} = \begin{bmatrix} q_{zc_{11}} & q_{zc_{12}} & \cdots & q_{zc_{1p}} \\ q_{zc_{21}} & q_{zc_{22}} & \cdots & q_{zc_{2p}} \end{bmatrix} \tag{VII.204}$$

where the elements of the matrix Q_{zP_1} take on different values depending on the type of bifurcation involved, as set out in Appendix D. Since the matrices Q_{zP_2}, Q_{zP_3} and Q_{zm_2} are rather complicated in general, and have only a limited effect on the outcome, we will detail them later.

Now we want to show how to calculate the coefficient matrices $Q_{cm}(\Pi_L)$ and $C_{cm}(\Pi_L, \Pi_Q)$, which appear in equation VII.195, for the case of two dimensional

bifurcations. The matrices $Q_{cm}(\Pi_L)$ and $C_{cm}(\Pi_L, \Pi_Q)$ are given by equations VII.41 and VII.42 respectively, which we restate here as

$$Q_{cm}(\Pi_L) = Q_{z_{P_1}} + Q_{z_{m_1}} M_1(\Pi_L) + Q_{z_c} M_2(\Pi_L) \tag{VII.205}$$

$$C_{cm}(\Pi_L, \Pi_Q) = \tilde{C}_z(\Pi_L) + Q_{z_{m_1}} N_1(\Pi_Q) + Q_{z_c} N_2(\Pi_L, \Pi_Q) \tag{VII.206}$$

where we will use equations VII.202, VII.203, and VII.204 and where we assume that the matrix $\tilde{C}_z(\Pi_L)$ will be calculated for the specific system being analyzed once Π_L is known. The two dimensional form of the matrices $\Pi_L \in R^{p\times 3}$ and $\Pi_Q \in R^{p\times 6}$ can be stated here as

$$\Pi_L = \begin{bmatrix} \Pi_{L_{11}} & \Pi_{L_{12}} & \Pi_{L_{13}} \\ \vdots & \vdots & \vdots \\ \Pi_{L_{p1}} & \Pi_{L_{p2}} & \Pi_{L_{p3}} \end{bmatrix} \tag{VII.207}$$

$$\Pi_Q = \begin{bmatrix} \Pi_{Q_{11}} & \Pi_{Q_{12}} & \Pi_{Q_{13}} & \Pi_{Q_{14}} & \Pi_{Q_{15}} & \Pi_{Q_{16}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Pi_{Q_{p1}} & \Pi_{Q_{p2}} & \Pi_{Q_{p3}} & \Pi_{Q_{p4}} & \Pi_{Q_{p5}} & \Pi_{Q_{p6}} \end{bmatrix} \tag{VII.208}$$

and the two dimensional form of the matrices $M_1(\Pi_L) \in R^{3\times 6}$, $M_2(\Pi_L) \in R^{p\times 6}$, $N_1(\Pi_Q) \in R^{3\times 10}$, and $N_2(\Pi_L, \Pi_Q) \in R^{p\times 10}$ are given in Appendix C by the formulas

$$M_1(\Pi_L) = \begin{bmatrix} \Pi_{L_{11}} & \Pi_{L_{12}} & \Pi_{L_{13}} & 0 & 0 & 0 \\ 0 & \Pi_{L_{11}} & 0 & \Pi_{L_{12}} & \Pi_{L_{13}} & 0 \\ 0 & 0 & \Pi_{L_{11}} & 0 & \Pi_{L_{12}} & \Pi_{L_{13}} \end{bmatrix} \tag{VII.209}$$

$$M_2(\Pi_L) = \begin{bmatrix} (\Pi_{L_{11}})^2 & 2\Pi_{L_{11}}\Pi_{L_{12}} & 2\Pi_{L_{11}}\Pi_{L_{13}} & (\Pi_{L_{12}})^2 & 2\Pi_{L_{12}}\Pi_{L_{13}} & (\Pi_{L_{13}})^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(\Pi_{L_{p1}}\right)^2 & 2\Pi_{L_{p1}}\Pi_{L_{p2}} & 2\Pi_{L_{p1}}\Pi_{L_{p3}} & \left(\Pi_{L_{p2}}\right)^2 & 2\Pi_{L_{p2}}\Pi_{L_{p3}} & \left(\Pi_{L_{p3}}\right)^2 \end{bmatrix} \tag{VII.210}$$

$$N_1(\Pi_Q) = \begin{bmatrix} \Pi_{Q_{11}} & \Pi_{Q_{12}} & \Pi_{Q_{13}} & \Pi_{Q_{14}} & \Pi_{Q_{15}} & \Pi_{Q_{16}} & 0 & 0 & 0 & 0 \\ 0 & \Pi_{Q_{11}} & 0 & \Pi_{Q_{12}} & \Pi_{Q_{13}} & 0 & \Pi_{Q_{14}} & \Pi_{Q_{15}} & \Pi_{Q_{16}} & 0 \\ 0 & 0 & \Pi_{Q_{11}} & 0 & \Pi_{Q_{12}} & \Pi_{Q_{13}} & 0 & \Pi_{Q_{14}} & \Pi_{Q_{15}} & \Pi_{Q_{16}} \end{bmatrix} \tag{VII.211}$$

and

$$N_2\left(\Pi_L, \Pi_Q\right) = \begin{bmatrix} n_{1_1} & n_{1_2} & n_{1_3} & n_{1_4} & n_{1_5} & n_{1_6} & n_{1_7} & n_{1_8} & n_{1_9} & n_{1_{10}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n_{p_1} & n_{p_2} & n_{p_3} & n_{p_4} & n_{p_5} & n_{p_6} & n_{p_7} & n_{p_8} & n_{p_9} & n_{p_{10}} \end{bmatrix} \tag{VII.212}$$

where the elements of the matrix $N_2\left(\Pi_L, \Pi_Q\right)$ are given by the formulas (for $i = 1$ to p)

$$\begin{aligned} n_{i_1} &= 2\Pi_{L_{i1}}\Pi_{Q_{i1}} \\ n_{i_2} &= 2\left(\Pi_{L_{i1}}\Pi_{Q_{i2}} + \Pi_{L_{i2}}\Pi_{Q_{i1}}\right) \\ n_{i_3} &= 2\left(\Pi_{L_{i1}}\Pi_{Q_{i3}} + \Pi_{L_{i3}}\Pi_{Q_{i1}}\right) \\ n_{i_4} &= 2\left(\Pi_{L_{i1}}\Pi_{Q_{i4}} + \Pi_{L_{i2}}\Pi_{Q_{i2}}\right) \\ n_{i_5} &= 2\left(\Pi_{L_{i1}}\Pi_{Q_{i5}} + \Pi_{L_{i2}}\Pi_{Q_{i3}} + \Pi_{L_{i2}}\Pi_{Q_{i3}}\right) \\ n_{i_6} &= 2\left(\Pi_{L_{i1}}\Pi_{Q_{i6}} + \Pi_{L_{i3}}\Pi_{Q_{i3}}\right) \\ n_{i_7} &= 2\Pi_{L_{i2}}\Pi_{Q_{i4}} \\ n_{i_8} &= 2\left(\Pi_{L_{i2}}\Pi_{Q_{i5}} + \Pi_{L_{i3}}\Pi_{Q_{i4}}\right) \\ n_{i_9} &= 2\left(\Pi_{L_{i2}}\Pi_{Q_{i6}} + \Pi_{L_{i3}}\Pi_{Q_{i5}}\right) \\ n_{i_{10}} &= 2\Pi_{L_{i3}}\Pi_{Q_{i6}} \end{aligned} \tag{VII.213}$$

Now, the elements of the matrices $Q_{cm}\left(\Pi_L\right)$ and $C_{cm}\left(\Pi_L, \Pi_Q\right)$ are much too complicated (when calculated out in full) to gain any insight into the general case. Instead, we adopt a different strategy. We state the form of the coefficient matrices, i.e.

$$Q_{cm}\left(\Pi_L\right) = \begin{bmatrix} q_{cm_{1_1}} & q_{cm_{1_2}} & q_{cm_{1_3}} & q_{cm_{1_4}} & q_{cm_{1_5}} & q_{cm_{1_6}} \\ q_{cm_{2_1}} & q_{cm_{2_2}} & q_{cm_{2_3}} & q_{cm_{2_4}} & q_{cm_{2_5}} & q_{cm_{2_6}} \end{bmatrix} \tag{VII.214}$$

and

$$C_{cm}\left(\Pi_L, \Pi_Q\right) = \begin{bmatrix} c_{cm_{1_1}} & c_{cm_{1_2}} & c_{cm_{1_3}} & c_{cm_{1_4}} & c_{cm_{1_5}} & c_{cm_{1_6}} & c_{cm_{1_7}} & c_{cm_{1_8}} & c_{cm_{1_9}} & c_{cm_{1_{10}}} \\ c_{cm_{2_1}} & c_{cm_{2_2}} & c_{cm_{2_3}} & c_{cm_{2_4}} & c_{cm_{2_5}} & c_{cm_{2_6}} & c_{cm_{2_7}} & c_{cm_{2_8}} & c_{cm_{2_9}} & c_{cm_{2_{10}}} \end{bmatrix} \tag{VII.215}$$

and we use analysis to determine which of the various coefficients have an effect on the dynamics. The Poincare normal forms attainable for each individual type of bifurcation (see Appendix D) result in formulas for the quadratic and cubic coefficients of the transformed system in terms of the quadratic and cubic coefficients of the original system. These formulas often depend on only a few of the original coefficients. Using these formulas, and then calculating only those elements of the matrices $Q_{cm}(\Pi_L)$ and $C_{cm}(\Pi_L, \Pi_Q)$ which are needed, is a more efficient method than calculating the elements for the general case, as we will see.

6. Hopf Bifurcations

Control systems in linear normal form with

$$F_z = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \tag{VII.216}$$

$(\omega_0 \neq 0)$, which implies

$$F_\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{VII.217}$$

experience a linearly unstabilizable Hopf bifurcation, as we will show. Hopf bifurcations are characterized by equations which, in polar coordinates, have the simplified form

$$\dot{r} = \alpha_1 \mu_1 r + a_0 r^3 + H.O.T. \tag{VII.218}$$

$$\dot{\theta} = \omega_0 + H.O.T. \tag{VII.219}$$

and which, when written out in full Poincare normal form, are

$$\dot{r} = \alpha(\mu_1) r + a(\mu_1) r^3 + O^{(4+)} \tag{VII.220}$$

$$\dot{\theta} = \omega(\mu_1) + b(\mu_1) r^2 + O^{(3+)} \tag{VII.221}$$

with

$$\alpha(\mu_1) = \alpha_1 \mu_1 + \alpha_2 \mu_1^2 + O^{(3+)} \tag{VII.222}$$

$$a(\mu_1) = a_0 + O^{(1+)} \tag{VII.223}$$

$$\omega(\mu_1) = \omega_0 + \omega_1\mu_1 + \omega_2\mu_1^2 + O^{(3+)} \tag{VII.224}$$

$$b(\mu_1) = b_0 + O^{(1+)} \tag{VII.225}$$

In order to analyze the system, we will be transforming our system into Cartesian coordinates. However, some important observations need to be made about equations VII.218 and VII.219 before we do so. First, we look for the existence, creation or destruction of equilibrium points. For $\omega_0 \neq 0$, the system appears to be devoid of local equilibrium points, since equation VII.219 requires $\dot{\theta}$ to be non-zero always. This observation is true, with one important exception: the nature of polar coordinates allows this system to have the possibility of a single equilibrium point if $\dot{r} = 0$ at the origin. Examining equation VII.218, we see that this condition is satisfied, i.e. that $\dot{r} = 0$ at the origin, and so the origin is the single local equilibrium point in the system. One equilibrium point exists, and none are created or destroyed. Second, we check for changes in the stability of the equilibrium point. The stability of the origin changes at $\mu_1 = 0$, which is our bifurcation point. For $\alpha_1 > 0$, the origin is stable when $\mu_1 < 0$, and unstable when $\mu_1 > 0$. For $\alpha_1 < 0$, the origin is stable when $\mu_1 > 0$, and unstable when $\mu_1 < 0$. Thus, the sign of α_1 determines on what side of the origin the bifurcation occurs. Third, we look for the existence, creation or destruction of limit cycles. Equation VII.218, taken by itself, indicates that a pitchfork bifurcation occurs in the r dynamics at the origin. However equation VII.219 precludes the existence of any local equilibrium points away from the origin. So, the pitchfork bifurcation in the r dynamics is actually indicative of the existence of a limit cycle at

$$r^* = \sqrt{\frac{-\alpha_1\mu_1}{a_0}} \tag{VII.226}$$

(for values of μ_1 such that r^* exists), such that the limit cycle is stable for $a_0 < 0$, and is unstable for $a_0 > 0$. So, we have four possible cases for the local dynamics at the origin:

- For $a_0 < 0$ and $\alpha_1 < 0$: when $\mu_1 < 0$, the origin is unstable, but is surrounded by a stable limit cycle; when $\mu_1 > 0$ no limit cycle around the origin exists, but the origin is stable.
- For $a_0 < 0$ and $\alpha_1 > 0$: when $\mu_1 < 0$, no limit cycle around the origin exists, but the origin is stable; when $\mu_1 > 0$, the origin is unstable, but is surrounded by a stable limit cycle.
- For $a_0 > 0$ and $\alpha_1 < 0$: when $\mu_1 < 0$, the origin is unstable and no limit cycle exists; when $\mu_1 > 0$, the origin is stable, but is surrounded by an unstable limit cycle.
- For $a_0 > 0$ and $\alpha_1 > 0$: when $\mu_1 < 0$, the origin is stable, but is surrounded by an unstable limit cycle; when $\mu_1 > 0$, the origin is unstable and no limit cycle around the origin exists.

Now we can transform our system to Cartesian coordinates. Using

$$z_1 = r\cos\theta \tag{VII.227}$$

$$z_2 = r\sin\theta \tag{VII.228}$$

equations VII.220 and VII.221 become

$$\dot{z}_1 = \alpha(\mu_1)z_1 - \omega(\mu_1)z_2 + (a(\mu_1)z_1 - b(\mu_1)z_2)\left(z_1^2 + z_2^2\right) + O^{(4+)} \tag{VII.229}$$

$$\dot{z}_2 = \omega(\mu_1)z_1 + \alpha(\mu_1)z_2 + (b(\mu_1)z_1 + a(\mu_1)z_2)\left(z_1^2 + z_2^2\right) + O^{(4+)} \tag{VII.230}$$

which expands out using equations VII.222, VII.223, VII.224 and VII.225 to become

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = F_{Hopf}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + Q_{Hopf}\begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ \mu_1 z_2 \\ z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix} + C_{Hopf}\begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1^2 z_2 \\ \mu_1 z_1^2 \\ \mu_1 z_1 z_2 \\ \mu_1 z_2^2 \\ z_1^3 \\ z_1^2 z_2 \\ z_1 z_2^2 \\ z_2^3 \end{bmatrix} + O^{(4+)} \tag{VII.231}$$

where

$$F_{Hopf} = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \tag{VII.232}$$

$$Q_{Hopf} = \begin{bmatrix} 0 & \alpha_1 & -\omega_1 & 0 & 0 & 0 \\ 0 & \omega_1 & \alpha_1 & 0 & 0 & 0 \end{bmatrix} \tag{VII.233}$$

$$C_{Hopf} = \begin{bmatrix} 0 & \alpha_2 & -\omega_2 & 0 & 0 & 0 & a_0 & -b_0 & a_0 & -b_0 \\ 0 & \omega_2 & \alpha_2 & 0 & 0 & 0 & b_0 & a_0 & b_0 & a_0 \end{bmatrix} \tag{VII.234}$$

So, Hopf bifurcations are characterized by equations of the form of (VII.231).

When a stable equilibrium point bifurcates into an unstable equilibrium point surrounded by a stable limit cycle, the Hopf bifurcation is said to be supercritical. When a stable equilibrium point surrounded by an unstable limit cycle bifurcates into a naked unstable equilibrium point, the Hopf bifurcation is said to be subcritical. For either type of Hopf bifurcation, trajectories in the vicinity of the origin are attracted when the origin is stable, and are repelled when the origin is unstable. For the supercritical case, trajectories diverging from the origin when it is unstable are captured by the stable limit cycle which surrounds it, so the magnitude of the excursion is limited. But for the subcritical case, trajectories diverging from the origin when it is unstable continue to diverge without limit (up to third order, anyway), and the magnitude of the excursion is unbounded, because there is nothing to "catch" them. Subcritical Hopf bifurcations create hysteresis in the system, which is often dangerous or damaging to real engineering systems. An example occurs in turbine engine compressors, where a subcritical Hopf bifurcation characterizes rotating stall and can cause engine flameout or compressor blade damage. The focus of subsequent sections will be to find ways to apply feedback to a system experiencing a linearly uncontrollable Hopf bifurcation to force the bifurcation to become supercritical. Examining the four cases detailed above, that means that we want to force the term a_0 to be less than zero. Along the way, we will discover that the application of feedback will also allow us to manipulate the values of α_1 and ω_1 to our advantage.

a. Desired Closed Loop Dynamics After Control has been Applied

For a Hopf bifurcation, the center manifold dynamics of a two-dimensional system are given by equation VII.195, and have the form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = F_z \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + Q_{cm}\left(\Pi_L\right) \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ \mu_1 z_2 \\ z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix} + C_{cm}\left(\Pi_L, \Pi_Q\right) \begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1^2 z_2 \\ \mu_1 z_1^2 \\ \mu_1 z_1 z_2 \\ \mu_1 z_2^2 \\ z_1^3 \\ z_1^2 z_2 \\ z_1 z_2^2 \\ z_2^3 \end{bmatrix} + O^{(4+)} \qquad \text{(VII.235)}$$

where

$$F_z = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \qquad \text{(VII.236)}$$

and where

$$F_\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(VII.237)}$$

is implicit in the equation. The matrices $Q_{cm}\left(\Pi_L\right)$ and $C_{cm}\left(\Pi_L, \Pi_Q\right)$ are taken from equations VII.214 and VII.215, respectively. The question is, for systems of the form of equation VII.235, when $\omega_0 \neq 0$, does the system experience a Hopf bifurcation at the origin? The answer is yes. Appendix D shows that a system of the form of equation VII.235 can be transformed by an appropriate coordinate transformation

into a system of the form

$$\begin{bmatrix} \dot{\hat{z}}_1 \\ \dot{\hat{z}}_2 \end{bmatrix} = F_z \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} + Q^*_{Hopf} \begin{bmatrix} \mu_1^2 \\ \mu_1 \hat{z}_1 \\ \mu_1 \hat{z}_2 \\ \hat{z}_1^2 \\ \hat{z}_1 \hat{z}_2 \\ \hat{z}_2^2 \end{bmatrix} + C^*_{Hopf} \begin{bmatrix} \mu_1^3 \\ \mu_1^2 \hat{z}_1 \\ \mu_1^2 \hat{z}_2 \\ \mu_1 \hat{z}_1^2 \\ \mu_1 \hat{z}_1 \hat{z}_2 \\ \mu_1 \hat{z}_2^2 \\ \hat{z}_1^3 \\ \hat{z}_1^2 \hat{z}_2 \\ \hat{z}_1 \hat{z}_2^2 \\ \hat{z}_2^3 \end{bmatrix} + O^{(4+)} \qquad \text{(VII.238)}$$

with

$$Q^*_{Hopf} = \begin{bmatrix} 0 & \alpha_1^* & -\omega_1^* & 0 & 0 & 0 \\ 0 & \omega_1^* & \alpha_1^* & 0 & 0 & 0 \end{bmatrix} \qquad \text{(VII.239)}$$

and

$$C_{Hopf} = \begin{bmatrix} 0 & \alpha_2^* & -\omega_2^* & 0 & 0 & 0 & a_0^* & -b_0^* & a_0^* & -b_0^* \\ 0 & \omega_2^* & \alpha_2^* & 0 & 0 & 0 & b_0^* & a_0^* & b_0^* & a_0^* \end{bmatrix} \qquad \text{(VII.240)}$$

which is clearly in the form of VII.231. Appendix D also shows that the quadratic coefficients α_1^* and ω_1^* can be written as functions of the elements of the coefficient matrix $Q_{cm}(\Pi_L)$, as

$$\alpha_1^* = \frac{1}{2}\left(q_{cm_{1_2}} + q_{cm_{2_3}}\right) \qquad \text{(VII.241)}$$

$$\omega_1^* = \frac{1}{2}\left(-q_{cm_{1_3}} + q_{cm_{2_2}}\right) \qquad \text{(VII.242)}$$

and that the cubic coefficient a_0^* can be written as a function of the elements of the coefficient matrices $Q_{cm}(\Pi_L)$ and $C_{cm}(\Pi_L, \Pi_Q)$, as

$$a_0^* = \frac{1}{8}\left(3c_{cm_{1_7}} + c_{cm_{1_9}} + c_{cm_{2_8}} + 3c_{cm_{2_{10}}}\right) + \tilde{a}_0 \qquad \text{(VII.243)}$$

with

$$\tilde{a}_0 = \frac{1}{8\omega_0}\left(q_{cm_{1_5}}\left(q_{cm_{1_4}} + q_{cm_{1_6}}\right) - q_{cm_{2_5}}\left(q_{cm_{2_4}} + q_{cm_{2_6}}\right) - 2q_{cm_{1_4}}q_{cm_{2_4}} + 2q_{cm_{1_6}}q_{cm_{2_6}}\right) \qquad \text{(VII.244)}$$

So, equations VII.241, VII.242 and VII.243 can be used to pick values of the elements of the coefficient matrices $Q_{cm}(\Pi_L)$ and $C_{cm}(\Pi_L, \Pi_Q)$ which produce desired values for α_1^*, ω_1^* and a_0^*. (We will not be concerned with the cubic coefficients α_2^*, ω_2^*, or b_0^*, as they are essentially higher order terms which do not affect our ability to control the system.) Calculating the appropriate elements $q_{cm_{ij}}$ and $c_{cm_{hk}}$ of the coefficient matrices $Q_{cm}(\Pi_L)$ and $C_{cm}(\Pi_L, \Pi_Q)$ in terms of Π_L and Π_Q will be the task of subsequent sections.

b. Determining the Linear Terms of the Center Manifold (Π_L), and the Linear Gains

Now we want to determine how to manipulate Π_L so as to force α_1^* and ω_1^* to take on desired values. Recalling the discussion in previous sections, we will not develop formulas for every coefficient in the matrix $Q_{cm}(\Pi_L)$, but rather for only those specific coefficients which affect the equivalent quadratic dynamics. Looking at equations VII.241 and VII.242, we need to calculate the coefficients $q_{cm_{12}}$, $q_{cm_{13}}$, $q_{cm_{22}}$ and $q_{cm_{23}}$, which we find from equations VII.205 and VII.214 yielding the formulas

$$q_{cm_{12}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(Q_{zP_1} + Q_{zm_1} M_1(\Pi_L) + Q_{zc} M_2(\Pi_L) \right) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{VII.245}$$

$$q_{cm_{13}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(Q_{zP_1} + Q_{zm_1} M_1(\Pi_L) + Q_{zc} M_2(\Pi_L) \right) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{VII.246}$$

$$q_{cm_{22}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \left(Q_{zP_1} + Q_{zm_1} M_1(\Pi_L) + Q_{zc} M_2(\Pi_L)\right) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{VII.247}$$

$$q_{cm_{23}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \left(Q_{zP_1} + Q_{zm_1} M_1(\Pi_L) + Q_{zc} M_2(\Pi_L)\right) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{VII.248}$$

Now, there are eight combinations which are common to these four equations. Plugging in from equations VII.203, VII.204, VII.209, and VII.210 we have

$$\begin{bmatrix} 1 & 0 \end{bmatrix} Q_{zm_1} = \begin{bmatrix} q_{zm_{11}} & q_{zm_{12}} & q_{zm_{13}} \end{bmatrix} \tag{VII.249}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} Q_{zm_1} = \begin{bmatrix} q_{zm_{21}} & q_{zm_{22}} & q_{zm_{23}} \end{bmatrix} \tag{VII.250}$$

and

$$\begin{bmatrix} 1 & 0 \end{bmatrix} Q_{zc} = \begin{bmatrix} q_{zc_{11}} & q_{zc_{12}} & \cdots & q_{zc_{1p}} \end{bmatrix} \tag{VII.251}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} Q_{zc} = \begin{bmatrix} q_{zc_{21}} & q_{zc_{22}} & \cdots & q_{zc_{2p}} \end{bmatrix} \tag{VII.252}$$

and

$$M_1(\Pi_L) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \Pi_{L_{12}} \\ \Pi_{L_{11}} \\ 0 \end{bmatrix} \tag{VII.253}$$

$$M_1\left(\Pi_L\right)\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}\Pi_{L_{13}}\\0\\\Pi_{L_{11}}\end{bmatrix} \tag{VII.254}$$

and

$$M_2\left(\Pi_L\right)\begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}=\begin{bmatrix}2\Pi_{L_{11}}\Pi_{L_{12}}\\\vdots\\2\Pi_{L_{p1}}\Pi_{L_{p2}}\end{bmatrix} \tag{VII.255}$$

$$M_2\left(\Pi_L\right)\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}2\Pi_{L_{11}}\Pi_{L_{13}}\\\vdots\\2\Pi_{L_{p1}}\Pi_{L_{p3}}\end{bmatrix} \tag{VII.256}$$

Now, for a Hopf bifurcation, there is one additional simplification which is worth noting for the last two combinations containing $M_2\left(\Pi_L\right)$. From the Linear Center Manifold Solution theorem in Chapter VI we know that the first row of the matrix Π_L determines all the other rows. Restating the relationship, where Π_{L_i} indicates the ith row of the matrix, and grouping the z_1 and z_2 coefficients for convenience, we have

$$\Pi_{L_1}=\left[\begin{array}{cc}\Pi_{L_{11}} & \left[\begin{array}{cc}\Pi_{L_{12}} & \Pi_{L_{13}}\end{array}\right]\end{array}\right] \tag{VII.257}$$

and, for $i=2$ to p,

$$\Pi_{L_i}=\left[\begin{array}{cc}\Pi_{L_{i1}} & \left[\begin{array}{cc}\Pi_{L_{i2}} & \Pi_{L_{i3}}\end{array}\right]\end{array}\right] \tag{VII.258}$$

where

$$\Pi_{L_i} = \Pi_{L_1} \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^{i-1} \tag{VII.259}$$

$$= \Pi_{L_1} \begin{bmatrix} 0 & 0 \\ F_z^{i-2} F_\mu & F_z^{i-1} \end{bmatrix} = \begin{bmatrix} 0 & \begin{bmatrix} \Pi_{L_{12}} & \Pi_{L_{13}} \end{bmatrix} F_z^{i-1} \end{bmatrix}$$

since $F_\mu = 0$ for a Hopf bifurcation. So, for $i = 2$ to p we have

$$\Pi_{L_{i1}} = 0 \tag{VII.260}$$

When equation VII.260 is plugged into equations VII.255 and VII.256, a dramatic simplification occurs. We get

$$M_2(\Pi_L) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\Pi_{L_{11}}\Pi_{L_{12}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{VII.261}$$

$$M_2(\Pi_L) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\Pi_{L_{11}}\Pi_{L_{13}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{VII.262}$$

Now, plugging equations VII.249 through VII.254, and equations VII.261 and VII.262, into equations VII.245 through VII.248, we get

$$q_{cm_{1_2}} = q_{zP_{12}} + q_{z_{m_{11}}}\Pi_{L_{12}} + q_{z_{m_{12}}}\Pi_{L_{11}} + 2q_{z_{c_{11}}}\Pi_{L_{11}}\Pi_{L_{12}} \tag{VII.263}$$

$$q_{cm_{1_3}} = q_{zP_{13}} + q_{z_{m_{11}}}\Pi_{L_{13}} + q_{z_{m_{13}}}\Pi_{L_{11}} + 2q_{z_{c_{11}}}\Pi_{L_{11}}\Pi_{L_{13}} \tag{VII.264}$$

$$q_{cm_{22}} = q_{zP_{22}} + q_{zm_{21}}\Pi_{L_{12}} + q_{zm_{22}}\Pi_{L_{11}} + 2q_{zc_{21}}\Pi_{L_{11}}\Pi_{L_{12}} \qquad \text{(VII.265)}$$

$$q_{cm_{23}} = q_{zP_{23}} + q_{zm_{21}}\Pi_{L_{13}} + q_{zm_{23}}\Pi_{L_{11}} + 2q_{zc_{21}}\Pi_{L_{11}}\Pi_{L_{13}} \qquad \text{(VII.266)}$$

which we can plug into equations VII.241 and VII.242 to get

$$\begin{aligned} \alpha_1^* &= \frac{1}{2}\left(q_{zP_{12}} + q_{zP_{23}} + \left(q_{zm_{12}} + q_{zm_{23}}\right)\Pi_{L_{11}}\right) \\ &+ \frac{1}{2}\left(\left(q_{zm_{11}} + 2q_{zc_{11}}\Pi_{L_{11}}\right)\Pi_{L_{12}} + \left(q_{zm_{21}} + 2q_{zc_{21}}\Pi_{L_{11}}\right)\Pi_{L_{13}}\right) \end{aligned} \qquad \text{(VII.267)}$$

$$\begin{aligned} \omega_1^* &= \frac{1}{2}\left(-q_{zP_{13}} + q_{zP_{22}} + \left(-q_{zm_{13}} + q_{zm_{22}}\right)\Pi_{L_{11}}\right) \\ &+ \frac{1}{2}\left(\left(q_{zm_{21}} + 2q_{zc_{21}}\Pi_{L_{11}}\right)\Pi_{L_{12}} - \left(q_{zm_{11}} + 2q_{zc_{11}}\Pi_{L_{11}}\right)\Pi_{L_{13}}\right) \end{aligned} \qquad \text{(VII.268)}$$

Finally, equations VII.267 and VII.268 can be arranged in matrix form to solve for $\Pi_{L_{12}}$ and $\Pi_{L_{13}}$, with $\Pi_{L_{11}}$ as a free variable. We get

$$\begin{aligned} &\begin{bmatrix} \left(q_{zm_{11}} + 2q_{zc_{11}}\Pi_{L_{11}}\right) & \left(q_{zm_{21}} + 2q_{zc_{21}}\Pi_{L_{11}}\right) \\ \left(q_{zm_{21}} + 2q_{zc_{21}}\Pi_{L_{11}}\right) & -\left(q_{zm_{11}} + 2q_{zc_{11}}\Pi_{L_{11}}\right) \end{bmatrix} \begin{bmatrix} \Pi_{L_{12}} \\ \Pi_{L_{13}} \end{bmatrix} \qquad \text{(VII.269)} \\ &= \begin{bmatrix} 2\alpha_1^* - \left(q_{zP_{12}} + q_{zP_{23}} + \left(q_{zm_{12}} + q_{zm_{23}}\right)\Pi_{L_{11}}\right) \\ 2\omega_1^* - \left(-q_{zP_{13}} + q_{zP_{22}} + \left(-q_{zm_{13}} + q_{zm_{22}}\right)\Pi_{L_{11}}\right) \end{bmatrix} \end{aligned}$$

We present the solution in the form of a theorem and a corollary.

Theorem C.9 (Hopf Bifurcation Linear Gains) *For a control system in the quadratic normal form of equation VII.194, with*

$$F_z = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \qquad \text{(VII.270)}$$

($\omega_0 \neq 0$) which implies

$$F_\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(VII.271)}$$

and with the linear state feedback gains K_y chosen so as to stabilize the linearly controllable states y, the linear state feedback gains K_{μ_1}, K_{z_1} and K_{z_2} which force the coefficient of the $\mu_1 r$ term in equation VII.220 to the desired value α_1^*, and which force the coefficient of the μ_1 term in equation VII.221 to the desired value ω_1^*, are given by

$$K_{\mu_1} = -K_{y_1}\Pi_{L_{11}} \qquad \text{(VII.272)}$$

$$K_{z_1} = G_A \Pi_{L_{12}} + G_B \Pi_{L_{13}} \tag{VII.273}$$

$$K_{z_2} = G_A \Pi_{L_{13}} - G_B \Pi_{L_{12}} \tag{VII.274}$$

with the coefficients $\Pi_{L_{11}}$, $\Pi_{L_{12}}$ and $\Pi_{L_{13}}$ determined by solving equation VII.269. The coefficients G_A and G_B are dependent on whether the dimension of the linearly controllable states, p, is even or odd. For p odd we get

$$G_A = \left(\sum_{i=0}^{\frac{p-1}{2}} (-1)^{i+1} (\omega_0)^{2i} K_{y_{2i+1}} \right) \tag{VII.275}$$

$$G_B = \left((-1)^{\frac{p-1}{2}} (\omega_0)^p + \sum_{i=0}^{\frac{p-1}{2}} (-1)^i (\omega_0)^{2i-1} K_{y_{2i}} \right) \tag{VII.276}$$

and for p even we get

$$G_A = \left((-1)^{\frac{p}{2}} (\omega_0)^p + \sum_{i=0}^{\frac{p-2}{2}} (-1)^{i+1} (\omega_0)^{2i} K_{y_{2i+1}} \right) \tag{VII.277}$$

$$G_B = \left(\sum_{i=0}^{\frac{p}{2}} (-1)^i (\omega_0)^{2i-1} K_{y_{2i}} \right) \tag{VII.278}$$

where we have used the definition $K_{y_0} = 0$.

Proof. From the Linear Center Manifold Solution theorem of Chapter VI we have

$$\begin{bmatrix} K_\mu^T & K_z^T \end{bmatrix} = \Pi_{L_1} \left(\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^p - \sum_{i=1}^{p} K_{y_i} \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^{i-1} \right) \tag{VII.279}$$

where we have used the definition $\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^0 = I$, and where $F_\mu = 0$ for a Hopf bifurcation. Because of the block diagonal nature of this equation with $F_\mu = 0$, this equation separates into two parts which can be solved individually, i.e.

$$K_{\mu_1} = -K_{y_1} \Pi_{L_{11}} \tag{VII.280}$$

and

$$\begin{bmatrix} K_{z_1} & K_{z_2} \end{bmatrix} = \begin{bmatrix} \Pi_{L_{12}} & \Pi_{L_{13}} \end{bmatrix} \left(F_z^p - \sum_{i=1}^{p} K_{y_i} F_z^{i-1} \right) \tag{VII.281}$$

Equation VII.280 proves equation VII.272. To solve equation VII.281 requires examining the structure of F_z. For a Hopf bifurcation, we have

$$F_z = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} = \omega_0 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{VII.282}$$

Looking at powers of the basic matrix, we have

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{VII.283}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^3 = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{VII.284}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{VII.285}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{VII.286}$$

which is a four-cycle repetition, compressible to a two-cycle matrix repetition if we include powers of negative one. That is,

$$F_z^j = (\omega_0)^j (-1)^{\frac{j-1}{2}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{VII.287}$$

for j odd, and

$$F_z^j = (\omega_0)^j (-1)^{\frac{j}{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{VII.288}$$

for j even, including the case $j = 0$, where we have used the definition $F_z^0 = I$. Now we can look at our complete term, which is

$$\begin{aligned} &\left(F_z^p - \sum_{i=1}^{p} K_{y_i} F_z^{i-1}\right) \qquad \text{(VII.289)} \\ = &\left(\sum_{i=0}^{\frac{p-1}{2}} (-1)^{i+1} (\omega_0)^{2i} K_{y_{2i+1}}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$+ \quad \left((-1)^{\frac{p-1}{2}} (\omega_0)^p + \sum_{i=0}^{\frac{p-1}{2}} (-1)^i (\omega_0)^{2i-1} K_{y_{2i}} \right) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

for p odd, and

$$\left(F_z^p - \sum_{i=1}^{p} K_{y_i} F_z^{i-1} \right) \tag{VII.290}$$

$$= \quad \left((-1)^{\frac{p}{2}} (\omega_0)^p + \sum_{i=0}^{\frac{p-2}{2}} (-1)^{i+1} (\omega_0)^{2i} K_{y_{2i+1}} \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$+ \quad \left(\sum_{i=0}^{\frac{p}{2}} (-1)^i (\omega_0)^{2i-1} K_{y_{2i}} \right) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

for p even, where we have used the definition $K_{y_0} = 0$. If we define

$$G_A = \left(\sum_{i=0}^{\frac{p-1}{2}} (-1)^{i+1} (\omega_0)^{2i} K_{y_{2i+1}} \right) \tag{VII.291}$$

$$G_B = \left((-1)^{\frac{p-1}{2}} (\omega_0)^p + \sum_{i=0}^{\frac{p-1}{2}} (-1)^i (\omega_0)^{2i-1} K_{y_{2i}} \right) \tag{VII.292}$$

for p odd, and

$$G_A = \left((-1)^{\frac{p}{2}} (\omega_0)^p + \sum_{i=0}^{\frac{p-2}{2}} (-1)^{i+1} (\omega_0)^{2i} K_{y_{2i+1}} \right) \tag{VII.293}$$

$$G_B = \left(\sum_{i=0}^{\frac{p}{2}} (-1)^i (\omega_0)^{2i-1} K_{y_{2i}} \right) \tag{VII.294}$$

for p even, then we can write

$$\begin{aligned} \left(F_z^p - \sum_{i=1}^{p} K_{y_i} F_z^{i-1} \right) &= G_A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + G_B \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} G_A & -G_B \\ G_B & G_A \end{bmatrix} \end{aligned} \tag{VII.295}$$

which lets us write equation VII.281 as

$$\begin{bmatrix} K_{z_1} & K_{z_2} \end{bmatrix} = \begin{bmatrix} \Pi_{L_{12}} & \Pi_{L_{13}} \end{bmatrix} \begin{bmatrix} G_A & -G_B \\ G_B & G_A \end{bmatrix} \tag{VII.296}$$

which proves equations VII.273 and VII.274, and completes the proof. ◁

Corollary C.10 *The coefficients $\Pi_{L_{11}}$, $\Pi_{L_{12}}$ and $\Pi_{L_{13}}$ which solve equation VII.269 and force the desired values α_1^* and ω_1^* are given by*

$$\Pi_{L_{11}} = arbitrary \tag{VII.297}$$

$$\begin{aligned}\Pi_{L_{12}} &= \frac{\left(q_{z_{m_{11}}} + 2q_{z_{c_{11}}}\Pi_{L_{11}}\right)}{\Delta_{\Pi_L}}\left(2\alpha_1^* - \left(q_{z_{P_{12}}} + q_{z_{P_{23}}} + \left(q_{z_{m_{12}}} + q_{z_{m_{23}}}\right)\Pi_{L_{11}}\right)\right) \\ &+ \frac{\left(q_{z_{m_{21}}} + 2q_{z_{c_{21}}}\Pi_{L_{11}}\right)}{\Delta_{\Pi_L}}\left(2\omega_1^* - \left(-q_{z_{P_{13}}} + q_{z_{P_{22}}} + \left(-q_{z_{m_{13}}} + q_{z_{m_{22}}}\right)\Pi_{L_{11}}\right)\right)\end{aligned} \tag{VII.298}$$

$$\begin{aligned}\Pi_{L_{13}} &= \frac{\left(q_{z_{m_{21}}} + 2q_{z_{c_{21}}}\Pi_{L_{11}}\right)}{\Delta_{\Pi_L}}\left(2\alpha_1^* - \left(q_{z_{P_{12}}} + q_{z_{P_{23}}} + \left(q_{z_{m_{12}}} + q_{z_{m_{23}}}\right)\Pi_{L_{11}}\right)\right) \\ &- \frac{\left(q_{z_{m_{11}}} + 2q_{z_{c_{11}}}\Pi_{L_{11}}\right)}{\Delta_{\Pi_L}}\left(2\omega_1^* - \left(-q_{z_{P_{13}}} + q_{z_{P_{22}}} + \left(-q_{z_{m_{13}}} + q_{z_{m_{22}}}\right)\Pi_{L_{11}}\right)\right)\end{aligned} \tag{VII.299}$$

when $\Delta_{\Pi_L} \neq 0$, with

$$\Delta_{\Pi_L} = \left(q_{z_{m_{11}}} + 2q_{z_{c_{11}}}\Pi_{L_{11}}\right)^2 + \left(q_{z_{m_{21}}} + 2q_{z_{c_{21}}}\Pi_{L_{11}}\right)^2 \tag{VII.300}$$

For all cases of the Hopf Bifurcation Linear Gains theorem such that $q_{z_{m_{11}}}^2 + q_{z_{m_{21}}}^2 \neq 0$, we may choose K_{μ_1} arbitrarily, including $K_{\mu_1} = 0$. For degenerate cases such that $q_{z_{m_{11}}}^2 + q_{z_{m_{21}}}^2 = 0$ (which implies that both $q_{z_{m_{11}}} = 0$ and $q_{z_{m_{21}}} = 0$) K_{μ_1} must be chosen such that $K_{\mu_1} \neq 0$ to allow linear state feedback to force the coefficient of the $\mu_1 r$ term in equation VII.220 to the desired value α_1^*, and/or to force the coefficient of the μ_1 term in equation VII.221 to the desired value ω_1^*.

Proof. The coefficients $\Pi_{L_{12}}$ and $\Pi_{L_{13}}$ which force the desired values α_1^* and ω_1^* are determined by inverting equation VII.269, which gives equations VII.298 and VII.299. This is possible when the quantity

$$\Delta_{\Pi_L} = \left(q_{z_{m_{11}}} + 2q_{z_{c_{11}}}\Pi_{L_{11}}\right)^2 + \left(q_{z_{m_{21}}} + 2q_{z_{c_{21}}}\Pi_{L_{11}}\right)^2 \tag{VII.301}$$

is non-zero. For all possible values of the free variable $\Pi_{L_{11}}$, the minimum magnitude $\Delta_{\Pi_{L\,\mathrm{min}}} = q_{z_{m_{11}}}^2 + q_{z_{m_{21}}}^2$ is obtained when $\Pi_{L_{11}} = 0$. If $\Delta_{\Pi_{L\,\mathrm{min}}} \neq 0$, then equation VII.269 is invertible regardless of the value of $\Pi_{L_{11}}$, so we can choose K_{μ_1} arbitrarily, including $K_{\mu_1} = 0$, since equation VII.272 is invertible. ($K_{y_1} \neq 0$, since the vector of linear state feedback gains K_y has been chosen to make the closed loop matrix $A + BK_y^T$ stable. If $K_{y_1} = 0$, then the closed loop matrix $A + BK_y^T$ has a zero

column, and thus a zero eigenvalue, violating the assumption of stability.) However, if $\Delta_{\Pi_{L\,\mathrm{min}}} = 0$, then equation VII.269 is not invertible for $\Pi_{L_{11}} = 0$, which means that $\Pi_{L_{11}}$ must be non-zero to force the desired values α_1^* and ω_1^*. Then, by equation VII.272, that means that we must choose K_{μ_1} non-zero, which completes the proof. $\triangleleft$

c. *Determining the Quadratic Terms of the Center Manifold* (Π_Q), *and the Quadratic Gains*

Now, knowing the required value of Π_L, we want to determine how to manipulate Π_Q so as to force the desired value a_0^*. Repeating equations VII.243 and VII.244 for convenience,

$$a_0^* = \frac{1}{8}\left(3c_{cm_{1_7}} + c_{cm_{1_9}} + c_{cm_{2_8}} + 3c_{cm_{2_{10}}}\right) + \tilde{a}_0 \tag{VII.302}$$

with

$$\tilde{a}_0 = \frac{1}{8\omega_0}\left(q_{cm_{1_5}}\left(q_{cm_{1_4}} + q_{cm_{1_6}}\right) - q_{cm_{2_5}}\left(q_{cm_{2_4}} + q_{cm_{2_6}}\right) - 2q_{cm_{1_4}}q_{cm_{2_4}} + 2q_{cm_{1_6}}q_{cm_{2_6}}\right) \tag{VII.303}$$

we see that we need to calculate the four cubic coefficients $c_{cm_{1_7}}$, $c_{cm_{1_9}}$, $c_{cm_{2_8}}$ and $c_{cm_{2_{10}}}$, and the six quadratic coefficients $q_{cm_{1_4}}$, $q_{cm_{1_5}}$, $q_{cm_{1_6}}$, $q_{cm_{2_4}}$, $q_{cm_{2_5}}$, and $q_{cm_{2_6}}$. Since we already know Π_L, we will start by calculating $\tilde{a}_0$. We find the six quadratic coefficients from equations VII.205 and VII.214 yielding the formulas

$$q_{cm_{1_4}} = \begin{bmatrix} 1 & 0 \end{bmatrix}\left(Q_{z_{P_1}} + Q_{z_{m_1}}M_1\left(\Pi_L\right) + Q_{z_c}M_2\left(\Pi_L\right)\right)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{VII.304}$$

$$q_{cm_{1_5}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \Big(Q_{z_{P_1}} + Q_{z_{m_1}} M_1\left(\Pi_L\right) + Q_{z_c} M_2\left(\Pi_L\right)\Big) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{VII.305}$$

$$q_{cm_{1_6}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \Big(Q_{z_{P_1}} + Q_{z_{m_1}} M_1\left(\Pi_L\right) + Q_{z_c} M_2\left(\Pi_L\right)\Big) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{VII.306}$$

$$q_{cm_{2_4}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \Big(Q_{z_{P_1}} + Q_{z_{m_1}} M_1\left(\Pi_L\right) + Q_{z_c} M_2\left(\Pi_L\right)\Big) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{VII.307}$$

$$q_{cm_{2_5}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \Big(Q_{z_{P_1}} + Q_{z_{m_1}} M_1\left(\Pi_L\right) + Q_{z_c} M_2\left(\Pi_L\right)\Big) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{VII.308}$$

$$q_{cm_{2_6}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \left(Q_{z_{P_1}} + Q_{z_{m_1}} M_1(\Pi_L) + Q_{z_c} M_2(\Pi_L)\right) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{VII.309}$$

To calculate these values, we use four combinations previously developed in equations VII.249, VII.250, VII.251 and VII.252, and develop six new combinations. Plugging in from equations VII.203, VII.204, VII.209, and VII.210 we have

$$M_1(\Pi_L) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \Pi_{L_{12}} \\ 0 \end{bmatrix} \tag{VII.310}$$

$$M_1(\Pi_L) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \Pi_{L_{13}} \\ \Pi_{L_{12}} \end{bmatrix} \tag{VII.311}$$

$$M_1(\Pi_L) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \Pi_{L_{13}} \end{bmatrix} \tag{VII.312}$$

and

$$M_2\left(\Pi_L\right)\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}\left(\Pi_{L_{12}}\right)^2\\ \vdots\\ \left(\Pi_{L_{p2}}\right)^2\end{bmatrix} \tag{VII.313}$$

$$M_2\left(\Pi_L\right)\begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}2\Pi_{L_{12}}\Pi_{L_{13}}\\ \vdots\\ 2\Pi_{L_{p2}}\Pi_{L_{p3}}\end{bmatrix} \tag{VII.314}$$

$$M_2\left(\Pi_L\right)\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}\left(\Pi_{L_{13}}\right)^2\\ \vdots\\ \left(\Pi_{L_{p3}}\right)^2\end{bmatrix} \tag{VII.315}$$

Now, we plug equations VII.249 through VII.252, and equations VII.310 and VII.315, into equations VII.304 through VII.309. Repeating equation VII.9 for Π_{L_i} for clarity, we get

$$\Pi_{L_i}=\Pi_{L_1}\begin{bmatrix}0 & 0\\ F_\mu & F_z\end{bmatrix}^{i-1} \tag{VII.316}$$

which gives

$$\begin{bmatrix}\Pi_{L_{i2}} & \Pi_{L_{i3}}\end{bmatrix}=\omega_0^{i-1}\begin{bmatrix}\Pi_{L_{12}} & \Pi_{L_{13}}\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^{i-1} \tag{VII.317}$$

since $F_\mu = 0$. Then we can calculate

$$q_{cm_{14}} = q_{zP_{14}} + q_{zm_{12}} \Pi_{L_{12}} + \sum_{i=1}^{p} q_{zc_{1i}} \left(\Pi_{L_{i2}}\right)^2 \tag{VII.318}$$

$$q_{cm_{15}} = q_{zP_{15}} + q_{zm_{12}} \Pi_{L_{13}} + q_{zm_{13}} \Pi_{L_{12}} + 2\sum_{i=1}^{p} q_{zc_{1i}} \Pi_{L_{i2}} \Pi_{L_{i3}} \tag{VII.319}$$

$$q_{cm_{16}} = q_{zP_{16}} + q_{zm_{13}} \Pi_{L_{13}} + \sum_{i=1}^{p} q_{zc_{1i}} \left(\Pi_{L_{i3}}\right)^2 \tag{VII.320}$$

$$q_{cm_{24}} = q_{zP_{24}} + q_{zm_{22}} \Pi_{L_{12}} + \sum_{i=1}^{p} q_{zc_{2i}} \left(\Pi_{L_{i2}}\right)^2 \tag{VII.321}$$

$$q_{cm_{25}} = q_{zP_{25}} + q_{zm_{22}} \Pi_{L_{13}} + q_{zm_{23}} \Pi_{L_{12}} + 2\sum_{i=1}^{p} q_{zc_{2i}} \Pi_{L_{i2}} \Pi_{L_{i3}} \tag{VII.322}$$

$$q_{cm_{26}} = q_{zP_{26}} + q_{zm_{23}} \Pi_{L_{13}} + \sum_{i=1}^{p} q_{zc_{2i}} \left(\Pi_{L_{i3}}\right)^2 \tag{VII.323}$$

where we have calculated equation VII.318 by combining equations VII.249 and VII.310 to get

$$\begin{bmatrix} 1 & 0 \end{bmatrix} Q_{zm_1} M_1 \left(\Pi_L\right) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = q_{zm_{12}} \Pi_{L_{12}} \tag{VII.324}$$

and by combining equations VII.251 and VII.313 to get

$$\begin{bmatrix} 1 & 0 \end{bmatrix} Q_{zc} M_2 \left(\Pi_L\right) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \sum_{i=1}^{p} q_{zc_{1i}} \left(\Pi_{L_{i2}}\right)^2 \tag{VII.325}$$

The rest of the terms in the equations follow with similar calculations. Finally, to calculate $\tilde{a}_0$, we can plug into equations VII.318 through VII.323 into equation VII.303, which we repeat here for clarity

$$\tilde{a}_0 = \frac{1}{8\omega_0}\left(q_{cm1_5}\left(q_{cm1_4} + q_{cm1_6}\right) - q_{cm2_5}\left(q_{cm2_4} + q_{cm2_6}\right) - 2q_{cm1_4}q_{cm2_4} + 2q_{cm1_6}q_{cm2_6}\right) \tag{VII.326}$$

Now that we have found $\tilde{a}_0$, we need to calculate the four cubic coefficients c_{cm1_7}, c_{cm1_9}, c_{cm2_8} and $c_{cm2_{10}}$. These are calculated from equations VII.206 and VII.215. Grouping terms appropriately yields the formulas

$$3c_{cm1_7} + c_{cm1_9} = \begin{bmatrix} 1 & 0 \end{bmatrix}\left(\tilde{C}_z\left(\Pi_L\right) + Q_{z_{m_1}}N_1\left(\Pi_Q\right) + Q_{z_c}N_2\left(\Pi_L, \Pi_Q\right)\right)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{VII.327}$$

and

$$c_{cm_{2_8}} + 3c_{cm_{2_{10}}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \left(\tilde{C}_z\left(\Pi_L\right) + Q_{z_{m_1}} N_1\left(\Pi_Q\right) + Q_{z_c} N_2\left(\Pi_L, \Pi_Q\right)\right) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} \tag{VII.328}$$

Now, there are four new combinations common to these two equations. Plugging in from equations VII.203, VII.204, VII.211, and VII.212 we have

$$N_1\left(\Pi_Q\right) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3\Pi_{Q_{14}} + \Pi_{Q_{16}} \\ \Pi_{Q_{15}} \end{bmatrix} \tag{VII.329}$$

$$N_1(\Pi_Q)\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\\3\end{bmatrix}=\begin{bmatrix}0\\ \Pi_{Q_{15}}\\ \Pi_{Q_{14}}+3\Pi_{Q_{16}}\end{bmatrix} \tag{VII.330}$$

and

$$N_2(\Pi_L,\Pi_Q)\begin{bmatrix}0\\0\\0\\0\\0\\0\\3\\0\\1\\0\end{bmatrix}=\begin{bmatrix}6\Pi_{L_{12}}\Pi_{Q_{14}}+2\Pi_{L_{13}}\Pi_{Q_{15}}+2\Pi_{L_{12}}\Pi_{Q_{16}}\\ \vdots\\ 6\Pi_{L_{p2}}\Pi_{Q_{p4}}+2\Pi_{L_{p3}}\Pi_{Q_{p5}}+2\Pi_{L_{p2}}\Pi_{Q_{p6}}\end{bmatrix} \tag{VII.331}$$

$$N_2\left(\Pi_L, \Pi_Q\right)\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\\3\end{bmatrix} = \begin{bmatrix}2\Pi_{L_{13}}\Pi_{Q_{14}} + 2\Pi_{L_{12}}\Pi_{Q_{15}} + 6\Pi_{L_{13}}\Pi_{Q_{16}}\\ \vdots \\ 2\Pi_{L_{p3}}\Pi_{Q_{p4}} + 2\Pi_{L_{p2}}\Pi_{Q_{p5}} + 6\Pi_{L_{p3}}\Pi_{Q_{p6}}\end{bmatrix} \qquad \text{(VII.332)}$$

The coefficients of $\tilde{C}_z\left(\Pi_L\right)$ are determined by evaluating the two dimensional case for equation VII.43, which is

$$\tilde{C}_z\left(\Pi_L\right)\begin{bmatrix}\mu_1^3\\ \mu_1^2 z_1\\ \mu_1^2 z_2\\ \mu_1 z_1^2\\ \mu_1 z_1 z_2\\ \mu_1 z_2^2\\ z_1^3\\ z_1^2 z_2\\ z_1 z_2^2\\ z_2^3\end{bmatrix} = \left(Q_{z_{P_2}} N_3\left(\Omega_Q\right) + Q_{z_{m_2}} N_5\left(\Pi_L, \Omega_Q\right)\right)\begin{bmatrix}\mu_1^3\\ \mu_1^2 z_1\\ \mu_1^2 z_2\\ \mu_1 z_1^2\\ \mu_1 z_1 z_2\\ \mu_1 z_2^2\\ z_1^3\\ z_1^2 z_2\\ z_1 z_2^2\\ z_2^3\end{bmatrix} \qquad \text{(VII.333)}$$
$$+ \; f_z^{(3)}\left(\mu_1, z_1, z_2, w_{cm}^{(1)}, y_{cm}^{(1)}\right) + g_z^{(2)}\left(\mu_1, z_1, z_2, w_{cm}^{(1)}, y_{cm}^{(1)}\right) v^{(1)}$$

which we write as

$$\tilde{C}_z\left(\Pi_L\right) = \begin{bmatrix}\tilde{c}_{z1_1} & \tilde{c}_{z1_2} & \tilde{c}_{z1_3} & \tilde{c}_{z1_4} & \tilde{c}_{z1_5} & \tilde{c}_{z1_6} & \tilde{c}_{z1_7} & \tilde{c}_{z1_8} & \tilde{c}_{z1_9} & \tilde{c}_{z1_{10}}\\ \tilde{c}_{z2_1} & \tilde{c}_{z2_2} & \tilde{c}_{z2_3} & \tilde{c}_{z2_4} & \tilde{c}_{z2_5} & \tilde{c}_{z2_6} & \tilde{c}_{z2_7} & \tilde{c}_{z2_8} & \tilde{c}_{z2_9} & \tilde{c}_{z2_{10}}\end{bmatrix} \qquad \text{(VII.334)}$$

Knowing Π_L, the coefficient matrix $\tilde{C}_z\left(\Pi_L\right)$ is most easily evaluated for a specific system, following the method of Appendix C. (It is very difficult to evaluate $\tilde{C}_z\left(\Pi_L\right)$

for the general case, since the dimensions m and p of the vectors w and y respectively are not specified.) Combining these with equations VII.249 through VII.252, and plugging into equation VII.243 for a_0^* and rearranging, we get

$$\begin{aligned} 8\left(a_0^* - \tilde{a}_0\right) &= 3\tilde{c}_{z_{17}} + \tilde{c}_{z_{19}} + \tilde{c}_{z_{28}} + 3\tilde{c}_{z_{210}} \\ &+ \Pi_{Q_1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3q_{zm_{12}} + q_{zm_{23}} \\ q_{zm_{13}} + q_{zm_{22}} \\ q_{zm_{12}} + 3q_{zm_{23}} \end{bmatrix} \\ &+ 2\sum_{i=1}^{p} \Pi_{Q_i} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3q_{zc_{1i}}\Pi_{L_{i2}} + q_{zc_{2i}}\Pi_{L_{i3}} \\ q_{zc_{1i}}\Pi_{L_{i3}} + q_{zc_{2i}}\Pi_{L_{i2}} \\ q_{zc_{1i}}\Pi_{L_{i2}} + 3q_{zc_{2i}}\Pi_{L_{i3}} \end{bmatrix} \end{aligned} \tag{VII.335}$$

Now, equation VII.10 gives the rows of the quadratic terms of the center manifold in terms of the first row, Π_{Q_1}, developed in Chapter VI, and repeated here for clarity

$$\Pi_{Q_i} = \Pi_{Q_1} D_\xi^{i-1} - \sum_{j=0}^{i-1} \Gamma_{z_j}\left(\Pi_L\right) D_\xi^{i-j-1} \tag{VII.336}$$

where we have used the definitions $D_\xi^0 = I$ and $\Gamma_{z_0}\left(\Pi_L\right) = 0$. To solve for the value of Π_{Q_1} which forces the desired value a_0^*, we can plug equation VII.336 into equation VII.335 and rearrange to get

$$\Pi_{Q_1}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 3q_{zm_{12}} + q_{zm_{23}} \\ q_{zm_{13}} + q_{zm_{22}} \\ q_{zm_{12}} + 3q_{zm_{23}} \end{bmatrix} + 2\sum_{i=1}^{p} D_\xi^{i-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3q_{zc_{1i}}\Pi_{L_{i2}} + q_{zc_{2i}}\Pi_{L_{i3}} \\ q_{zc_{1i}}\Pi_{L_{i3}} + q_{zc_{2i}}\Pi_{L_{i2}} \\ q_{zc_{1i}}\Pi_{L_{i2}} + 3q_{zc_{2i}}\Pi_{L_{i3}} \end{bmatrix}\right) \tag{VII.337}$$

$$= \quad 8\left(a_0^* - \tilde{a}_0\right) - \left(3\tilde{c}_{z1_7} + \tilde{c}_{z1_9} + \tilde{c}_{z2_8} + 3\tilde{c}_{z2_{10}}\right)$$

$$+ \quad 2\sum_{i=1}^{p}\sum_{j=0}^{i-1}\Gamma_{z_j}\left(\Pi_L\right)D_\xi^{i-j-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 3q_{zc_{1_i}}\Pi_{L_{i2}} + q_{zc_{2_i}}\Pi_{L_{i3}} \\ q_{zc_{1_i}}\Pi_{L_{i3}} + q_{zc_{2_i}}\Pi_{L_{i2}} \\ q_{zc_{1_i}}\Pi_{L_{i2}} + 3q_{zc_{2_i}}\Pi_{L_{i3}} \end{bmatrix}$$

Equation VII.337 can now be used to find a solution for the value of Π_{Q_1} which forces the desired value a_0^*. However, first there is one more simplification worth making. The matrix D_ξ is block diagonal for a Hopf bifurcation, and given in Appendix A in block form as

$$D_\xi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & F_z & 0 \\ 0 & 0 & D_{zz} \end{bmatrix} \tag{VII.338}$$

This can be plugged into equation VII.337 to give

$$\Pi_{Q_1}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 3q_{z_{m_{12}}} + q_{z_{m_{23}}} \\ q_{z_{m_{13}}} + q_{z_{m_{22}}} \\ q_{z_{m_{12}}} + 3q_{z_{m_{23}}} \end{bmatrix} + 2\sum_{i=1}^{p}\begin{bmatrix} 0 \\ 0 \\ 0 \\ D_{zz}^{i-1}\begin{bmatrix} 3q_{zc_{1_i}}\Pi_{L_{i2}} + q_{zc_{2_i}}\Pi_{L_{i3}} \\ q_{zc_{1_i}}\Pi_{L_{i3}} + q_{zc_{2_i}}\Pi_{L_{i2}} \\ q_{zc_{1_i}}\Pi_{L_{i2}} + 3q_{zc_{2_i}}\Pi_{L_{i3}} \end{bmatrix} \end{bmatrix}\right) \tag{VII.339}$$

$$= \quad 8\left(a_0^* - \tilde{a}_0\right) - \left(3\tilde{c}_{z1_7} + \tilde{c}_{z1_9} + \tilde{c}_{z2_8} + 3\tilde{c}_{z2_{10}}\right)$$

$$+ \quad 2\sum_{i=1}^{p}\sum_{j=0}^{i-1}\Gamma_{z_j}\left(\Pi_L\right)\begin{bmatrix} 0 \\ 0 \\ 0 \\ D_{zz}^{i-j-1}\begin{bmatrix} 3q_{zc_{1_i}}\Pi_{L_{i2}} + q_{zc_{2_i}}\Pi_{L_{i3}} \\ q_{zc_{1_i}}\Pi_{L_{i3}} + q_{zc_{2_i}}\Pi_{L_{i2}} \\ q_{zc_{1_i}}\Pi_{L_{i2}} + 3q_{zc_{2_i}}\Pi_{L_{i3}} \end{bmatrix} \end{bmatrix}$$

which makes clear that only the last three components $\Pi_{Q_{14}}$, $\Pi_{Q_{15}}$ and $\Pi_{Q_{16}}$ of Π_{Q_1} have any influence on the solution. Now, since equation VII.339 has the form of a row vector times a column vector equalling a scalar value, that is

$$\Pi_{Q_1}V = \sigma \tag{VII.340}$$

with

$$V = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3q_{z_{m_{12}}} + q_{z_{m_{23}}} \\ q_{z_{m_{13}}} + q_{z_{m_{22}}} \\ q_{z_{m_{12}}} + 3q_{z_{m_{23}}} \end{bmatrix} + 2\sum_{i=1}^{p} \begin{bmatrix} 0 \\ 0 \\ 0 \\ D_{zz}^{i-1} \begin{bmatrix} 3q_{z_{c_{1_i}}} \Pi_{L_{i2}} + q_{z_{c_{2_i}}} \Pi_{L_{i3}} \\ q_{z_{c_{1_i}}} \Pi_{L_{i3}} + q_{z_{c_{2_i}}} \Pi_{L_{i2}} \\ q_{z_{c_{1_i}}} \Pi_{L_{i2}} + 3q_{z_{c_{2_i}}} \Pi_{L_{i3}} \end{bmatrix} \end{bmatrix} \qquad \text{(VII.341)}$$

and

$$\begin{aligned} \sigma &= 8\left(a_0^* - \tilde{a}_0\right) - \left(3\tilde{c}_{z_{1_7}} + \tilde{c}_{z_{1_9}} + \tilde{c}_{z_{2_8}} + 3\tilde{c}_{z_{2_{10}}}\right) \qquad \text{(VII.342)} \\ &+ 2\sum_{i=1}^{p}\sum_{j=0}^{i-1} \Gamma_{z_j}\left(\Pi_L\right) \begin{bmatrix} 0 \\ 0 \\ 0 \\ D_{zz}^{i-j-1} \begin{bmatrix} 3q_{z_{c_{1_i}}} \Pi_{L_{i2}} + q_{z_{c_{2_i}}} \Pi_{L_{i3}} \\ q_{z_{c_{1_i}}} \Pi_{L_{i3}} + q_{z_{c_{2_i}}} \Pi_{L_{i2}} \\ q_{z_{c_{1_i}}} \Pi_{L_{i2}} + 3q_{z_{c_{2_i}}} \Pi_{L_{i3}} \end{bmatrix} \end{bmatrix} \end{aligned}$$

we can pick any Π_{Q_1} whose projection onto the vector V has the value σ. The "most efficient" choice for Π_{Q_1}, and the one we will use, is when Π_{Q_1} points in the same direction as V and has length scaled to produce σ, that is

$$\Pi_{Q_1} = \left(\frac{\sigma}{V^T V}\right) V^T \qquad \text{(VII.343)}$$

This will be our solution method for the case of a Hopf bifurcation.

Before we proceed to a solution, it is worth our time to consider the matrices $\Gamma_z\left(\Pi_L\right)$ and Γ_K^T, which are used in the calculation of the scalar σ and the quadratic gain vector $\tilde{K}_{(2)}$. Restating the equations for both matrices from Chapter VI, we have

$$\Gamma_z\left(\Pi_L\right) = Q_{y_c} M_2\left(\Pi_L\right) - \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{z_{P_1}} + Q_{z_{m_1}} M_1\left(\Pi_L\right) + Q_{z_c} M_2\left(\Pi_L\right)\right) \qquad \text{(VII.344)}$$

and

$$\Gamma_K^T = -\sum_{j=0}^{p} \Gamma_{z_j}(\Pi_L) D_\xi^{p-j} + \sum_{i=1}^{p} K_{y_i} \sum_{j=0}^{i-1} \Gamma_{z_j}(\Pi_L) D_\xi^{i-j-1} \tag{VII.345}$$

where the rows of $\Gamma_z(\Pi_L)$ are used to calculate the matrix Γ_K^T, and where the matrix D_ξ is given in Appendix A, the matrices $M_1(\Pi_L)$ and $M_2(\Pi_L)$ are given in Appendix C, the matrices Q_{y_c}, $Q_{z_{P_1}}$, $Q_{z_{m_1}}$ and Q_{z_c} come from equation VII.5 and VII.194, and where we have used the definitions $D_\xi^0 = I$ and $\Gamma_{z_0}(\Pi_L) = 0$. By exploiting the structure of the matrix D_ξ, we will show that the matrix Γ_K^T simplifies for a Hopf bifurcation. We state our results in a lemma.

Lemma C.11 (Hopf Γ_K Matrix) *For a Hopf bifurcation, with the matrix $\Gamma_z(\Pi_L)$ defined in equation VII.344 given by*

$$\Gamma_z(\Pi_L) = \begin{bmatrix} \Gamma_{z_{11}}(\Pi_L) & \cdots & \Gamma_{z_{1p}}(\Pi_L) \\ \vdots & & \vdots \\ \Gamma_{z_{p1}}(\Pi_L) & \cdots & \Gamma_{z_{pp}}(\Pi_L) \end{bmatrix} \tag{VII.346}$$

then the matrix Γ_K^T defined in equation VII.345 and given by

$$\Gamma_K^T = \begin{bmatrix} \Gamma_{K_1} & \Gamma_{K_2} & \Gamma_{K_3} & \Gamma_{K_4} & \Gamma_{K_5} & \Gamma_{K_6} \end{bmatrix} \tag{VII.347}$$

separates to yield three block equations which can be solved individually

$$\Gamma_{K_1} = -\Gamma_{z_{p1}}(\Pi_L) + \sum_{j=0}^{p-1} K_{y_{j+1}} \Gamma_{z_{j1}}(\Pi_L) \tag{VII.348}$$

$$\begin{aligned} \begin{bmatrix} \Gamma_{K_2} & \Gamma_{K_3} \end{bmatrix} &= -\sum_{j=0}^{p} \begin{bmatrix} \Gamma_{z_{j2}}(\Pi_L) & \Gamma_{z_{j3}}(\Pi_L) \end{bmatrix} F_z^{p-j} \\ &+ \sum_{i=1}^{p} K_{y_i} \sum_{j=0}^{i-1} \begin{bmatrix} \Gamma_{z_{j2}}(\Pi_L) & \Gamma_{z_{j3}}(\Pi_L) \end{bmatrix} F_z^{i-j-1} \end{aligned} \tag{VII.349}$$

$$\begin{aligned} \begin{bmatrix} \Gamma_{K_4} & \Gamma_{K_5} & \Gamma_{K_6} \end{bmatrix} &= -\sum_{j=0}^{p} \begin{bmatrix} \Gamma_{z_{j4}}(\Pi_L) & \Gamma_{z_{j5}}(\Pi_L) & \Gamma_{z_{j6}}(\Pi_L) \end{bmatrix} D_{zz}^{p-j} \\ &+ \sum_{i=1}^{p} K_{y_i} \sum_{j=0}^{i-1} \begin{bmatrix} \Gamma_{z_{j4}}(\Pi_L) & \Gamma_{z_{j5}}(\Pi_L) & \Gamma_{z_{j6}}(\Pi_L) \end{bmatrix} D_{zz}^{i-j-1} \end{aligned} \tag{VII.350}$$

where the matrix D_{zz} is defined in Appendix A, and where we have used the definition $\Gamma_{z_0}(\Pi_L) = 0$.

Proof. Looking at the block structure of D_ξ from Appendix A we have

$$D_\xi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & F_z & 0 \\ 0 & 0 & D_{zz} \end{bmatrix} \tag{VII.351}$$

Plugging into equation VII.345 gives

$$\begin{aligned} \Gamma_K^T &= -\sum_{j=0}^{p} \Gamma_{z_j}(\Pi_L) \begin{bmatrix} 0^{p-j} & 0 & 0 \\ 0 & F_z^{p-j} & 0 \\ 0 & 0 & D_{zz}^{p-j} \end{bmatrix} \\ &+ \sum_{i=1}^{p} K_{y_i} \sum_{j=0}^{i-1} \Gamma_{z_j}(\Pi_L) \begin{bmatrix} 0^{i-j-1} & 0 & 0 \\ 0 & F_z^{i-j-1} & 0 \\ 0 & 0 & D_{zz}^{i-j-1} \end{bmatrix} \end{aligned} \tag{VII.352}$$

where we have used the definitions $0^j = I$ for $j = 0$ and $0^j = 0$ for $j \neq 0$. Then, partitioning the matrix Γ_K^T and the rows $\Gamma_{z_j}(\Pi_L)$ as given in the lemma, and multiplying out gives the expected result. ◁

This brings us to the Hopf bifurcation quadratic gains theorem.

Theorem C.12 (Hopf Quadratic Gains) *For a control system in the quadratic normal form of equation VII.194, with*

$$F_z = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \tag{VII.353}$$

($\omega_0 \neq 0$) which implies

$$F_\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{VII.354}$$

with the linear state feedback gains K_y chosen so as to stabilize the linearly controllable states y, and with the linear state feedback gains K_w and quadratic state feedback gains $K_{\mu y^{(2)}}$, $K_{zy^{(2)}}$ and $K_{y^{(2)}}$ set to zero, the quadratic state feedback gains which force the coefficient of the r^3 term in equation VII.220 to the desired value a_0^*, are given by

$$K_{z_1^2} = \left(-\frac{1}{2}G_C - K_{y_1}\right)\Pi_{Q_{14}} + \left(\frac{1}{2}G_D\right)\Pi_{Q_{15}} + \left(\frac{1}{2}G_C\right)\Pi_{Q_{16}} + \Gamma_{K_4} \tag{VII.355}$$

$$K_{z_1 z_2} = -(G_D)\Pi_{Q_{14}} + (-G_C - K_{y_1})\Pi_{Q_{15}} + (G_D)\Pi_{Q_{16}} + \Gamma_{K_5} \tag{VII.356}$$

$$K_{z_2^2} = \left(\frac{1}{2}G_C\right)\Pi_{Q_{14}} - \left(\frac{1}{2}G_D\right)\Pi_{Q_{15}} + \left(-\frac{1}{2}G_C - K_{y_1}\right)\Pi_{Q_{16}} + \Gamma_{K_6} \quad \text{(VII.357)}$$

with the coefficients $\Pi_{Q_{14}}$, $\Pi_{Q_{14}}$ and $\Pi_{Q_{14}}$ determined by solving equation VII.339. The formulas for the coefficients G_C and G_D depend on whether the dimension of the controllable states p is even or odd. For p odd we have

$$G_C = -\left(\sum_{i=0}^{\frac{p-1}{2}} (-1)^{i+1} (2\omega_0)^{2i} \tilde{K}_{y_{2i+1}}\right) \quad \text{(VII.358)}$$

$$G_D = \left((-1)^{\frac{p-1}{2}} (2\omega_0)^p - \sum_{i=0}^{\frac{p-1}{2}} (-1)^{i+1} (2\omega_0)^{2i-1} \tilde{K}_{y_{2i}}\right) \quad \text{(VII.359)}$$

and for p even we have

$$G_C = \left((-1)^{\frac{p-2}{2}} (2\omega_0)^p - \sum_{i=1}^{\frac{p}{2}} (-1)^{i} (2\omega_0)^{2i-2} \tilde{K}_{y_{2i-1}}\right) \quad \text{(VII.360)}$$

$$G_D = -\left(\sum_{i=1}^{\frac{p}{2}} (-1)^{i-1} (2\omega_0)^{2i-1} \tilde{K}_{y_{2i}}\right) \quad \text{(VII.361)}$$

where in both cases we have used the notation $\tilde{K}_{y_0} = 0$, $\tilde{K}_{y_1} = 0$, and $\tilde{K}_{y_j} = K_{y_j}$ for $j \neq 0, 1$. The coefficients Γ_{K_4}, Γ_{K_5} and Γ_{K_6} are given by equation VII.350 in the Hopf Γ_K Matrix lemma. The remaining quadratic state feedback gains $K_{\mu_1^2}$, $K_{\mu_1 z_1}$ and $K_{\mu_1 z_2}$ have no effect on the equivalent cubic order dynamics and may be chosen arbitrarily, including being set to zero.

Proof. From the Quadratic Center Manifold Solution theorem of Chapter VI we have

$$\begin{bmatrix} K^T_{\mu^{(2)}} & K^T_{\mu z^{(2)}} & K^T_{z^{(2)}} \end{bmatrix} = \Pi_{Q_1} D_K + \Gamma_K^T \quad \text{(VII.362)}$$

where

$$D_K = D_\xi^p - \sum_{i=1}^{p} K_{y_i} D_\xi^{i-1} \quad \text{(VII.363)}$$

and where we have used the definition $D_\xi^0 = I$. From Appendix A, for a Hopf bifurcation, the matrix D_ξ is given in block form by

$$D_\xi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & F_z & 0 \\ 0 & 0 & D_{zz} \end{bmatrix} \quad \text{(VII.364)}$$

so D_K can be expressed in block form as

$$D_K = \begin{bmatrix} -K_{y_1} & 0 & 0 \\ 0 & F_z^p - \sum_{i=1}^p K_{y_i} F_z^{i-1} & 0 \\ 0 & 0 & D_{zz}^p - \sum_{i=1}^p K_{y_i} D_{zz}^{i-1} \end{bmatrix} \qquad \text{(VII.365)}$$

where the matrix D_{zz} is given in Appendix A, and where we have used the definitions $F_z^0 = I$ and $D_{zz}^0 = I$. Following the example of the preceding lemma, we separate Γ_K^T into three corresponding pieces, and we end up with three block equations to solve for the Hopf quadratic gains, which are

$$K_{\mu_1^2} = -K_{y_1}\Pi_{Q_{11}} + \Gamma_{K_1} \qquad \text{(VII.366)}$$

$$\begin{aligned} \begin{bmatrix} K_{\mu_1 z_1} & K_{\mu_1 z_2} \end{bmatrix} &= \begin{bmatrix} \Pi_{Q_{12}} & \Pi_{Q_{13}} \end{bmatrix} \left(F_z^p - \sum_{i=1}^p K_{y_i} F_z^{i-1} \right) \\ &+ \begin{bmatrix} \Gamma_{K_2} & \Gamma_{K_3} \end{bmatrix} \end{aligned} \qquad \text{(VII.367)}$$

$$\begin{aligned} \begin{bmatrix} K_{z_1^2} & K_{z_1 z_2} & K_{z_2^2} \end{bmatrix} &= \begin{bmatrix} \Pi_{Q_{14}} & \Pi_{Q_{15}} & \Pi_{Q_{16}} \end{bmatrix} \left(D_{zz}^p - \sum_{i=1}^p K_{y_i} D_{zz}^{i-1} \right) \\ &+ \begin{bmatrix} \Gamma_{K_4} & \Gamma_{K_5} & \Gamma_{K_6} \end{bmatrix} \end{aligned} \qquad \text{(VII.368)}$$

Now, by inspection of equation VII.339, the only components of Π_{Q_1} which can affect a_0^*, our desired cubic coefficient, are $\Pi_{Q_{14}}$, $\Pi_{Q_{15}}$ and $\Pi_{Q_{16}}$. Therefore, we have to solve equation VII.368 for our quadratic gains, since it alone contains $\Pi_{Q_{14}}$, $\Pi_{Q_{15}}$ and/or $\Pi_{Q_{16}}$. From Appendix A, for a Hopf bifurcation, we have

$$D_{zz} = \begin{bmatrix} 0 & -2\omega_0 & 0 \\ \omega_0 & 0 & -\omega_0 \\ 0 & 2\omega_0 & 0 \end{bmatrix} = 2\omega_0 \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{(VII.369)}$$

Looking at powers of the basic matrix, we have

$$\begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \qquad \text{(VII.370)}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}^3 = -\begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \tag{VII.371}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}^4 = -\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \tag{VII.372}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}^5 = \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \tag{VII.373}$$

which is a four-cycle repetition, compressible to a two-cycle matrix repetition if we include powers of negative one. That is,

$$D^j_{zz} = (2\omega_0)^j \,(-1)^{\frac{j-1}{2}} \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \tag{VII.374}$$

for j odd, and

$$D^j_{zz} = (2\omega_0)^j \,(-1)^{\frac{j-2}{2}} \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \tag{VII.375}$$

for $j \geq 2$ even, and where we have again used the definition

$$D^0_{zz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{VII.376}$$

Now we can look at our complete term, which is

$$\left(D^p_{zz} - \sum_{i=1}^{p} K_{y_i} D^{i-1}_{zz} \right) \tag{VII.377}$$

$$= -\left(\sum_{i=0}^{\frac{p-1}{2}} (-1)^{i+1} \,(2\omega_0)^{2i}\, \tilde{K}_{y_{2i+1}} \right) \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

$$+ \left((-1)^{\frac{p-1}{2}} (2\omega_0)^p - \sum_{i=0}^{\frac{p-1}{2}} (-1)^{i+1} (2\omega_0)^{2i-1} \tilde{K}_{y_{2i}} \right) \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$- \; K_{y_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for p odd, and

$$\left(D_{zz}^p - \sum_{i=1}^{p} K_{y_i} D_{zz}^{i-1} \right) \tag{VII.378}$$

$$= \left((-1)^{\frac{p-2}{2}} (2\omega_0)^p - \sum_{i=1}^{\frac{p}{2}} (-1)^{i} (2\omega_0)^{2i-2} \tilde{K}_{y_{2i-1}} \right) \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

$$- \left(\sum_{i=1}^{\frac{p}{2}} (-1)^{i-1} (2\omega_0)^{2i-1} \tilde{K}_{y_{2i}} \right) \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$- \; K_{y_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for p even, where we have used the definitions $\tilde{K}_{y_0} = \tilde{K}_{y_1} = 0$ and $\tilde{K}_{y_j} = K_{y_j}$ for $j \neq 0, 1$. Now, if we define

$$G_C = - \left(\sum_{i=0}^{\frac{p-1}{2}} (-1)^{i+1} (2\omega_0)^{2i} \tilde{K}_{y_{2i+1}} \right) \tag{VII.379}$$

$$G_D = \left((-1)^{\frac{p-1}{2}} (2\omega_0)^p - \sum_{i=0}^{\frac{p-1}{2}} (-1)^{i+1} (2\omega_0)^{2i-1} \tilde{K}_{y_{2i}} \right) \tag{VII.380}$$

for p odd, and

$$G_C = \left((-1)^{\frac{p-2}{2}} (2\omega_0)^p - \sum_{i=1}^{\frac{p}{2}} (-1)^{i} (2\omega_0)^{2i-2} \tilde{K}_{y_{2i-1}} \right) \tag{VII.381}$$

$$G_D = -\left(\sum_{i=1}^{\frac{p}{2}} (-1)^{i-1} (2\omega_0)^{2i-1} \tilde{K}_{y_{2i}}\right) \tag{VII.382}$$

for p even, then we can write

$$\begin{aligned}
&\left(D_{zz}^p - \sum_{i=1}^{p} K_{y_i} D_{zz}^{i-1}\right) \qquad \text{(VII.383)} \\
&= G_C \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} + G_D \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} - K_{y_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2}G_C - K_{y_1} & -G_D & \frac{1}{2}G_C \\ \frac{1}{2}G_D & -G_C - K_{y_1} & -\frac{1}{2}G_D \\ \frac{1}{2}G_C & G_D & -\frac{1}{2}G_C - K_{y_1} \end{bmatrix}
\end{aligned}$$

which lets us write equation VII.368 as

$$\begin{aligned}
&\begin{bmatrix} K_{z_1^2} & K_{z_1 z_2} & K_{z_2^2} \end{bmatrix} \qquad \text{(VII.384)} \\
&= \begin{bmatrix} \Pi_{Q_{14}} & \Pi_{Q_{15}} & \Pi_{Q_{16}} \end{bmatrix} \begin{bmatrix} -\frac{1}{2}G_C - K_{y_1} & -G_D & \frac{1}{2}G_C \\ \frac{1}{2}G_D & -G_C - K_{y_1} & -\frac{1}{2}G_D \\ \frac{1}{2}G_C & G_D & -\frac{1}{2}G_C - K_{y_1} \end{bmatrix} \\
&+ \begin{bmatrix} \Gamma_{K_4} & \Gamma_{K_5} & \Gamma_{K_6} \end{bmatrix}
\end{aligned}$$

which proves the first part of the theorem. Because the terms $\Pi_{Q_{11}}$, $\Pi_{Q_{12}}$ and $\Pi_{Q_{13}}$ have no influence in equation VII.339 on forcing the coefficient a_0^* to the desired value, they can be chosen arbitrarily. By equation VII.366, this means that the quadratic gain $K_{\mu_1^2}$ can be chosen arbitrarily, including being set to zero, since $K_{y_1} \neq 0$ by the arguments given in the proof of the Hopf Linear Gains theorem. By equation VII.367, this means that the quadratic gains $K_{\mu_1 z_1}$ and $K_{\mu_1 z_2}$ can be chosen arbitrarily, including being set to zero, since the matrix $(F_z^p - \sum_{i=1}^p K_{y_i} F_z^{i-1})$ is invertible, which proves the remainder of the theorem. ◁

Corollary C.13 (Hopf Quadratic Gains Linear Zero) *For the special case of $K_{\mu_1} = K_{z_1} = K_{z_2} = 0$, the Hopf Quadratic Gains theorem may be solved using*

the coefficients $\Pi_{Q_{14}}$, $\Pi_{Q_{14}}$ *and* $\Pi_{Q_{14}}$ *given by*

$$\Pi_{Q_{14}} = G_E\left(3q_{zm_{12}} + q_{zm_{23}}\right) \tag{VII.385}$$

$$\Pi_{Q_{15}} = G_E\left(q_{zm_{13}} + q_{zm_{22}}\right) \tag{VII.386}$$

$$\Pi_{Q_{16}} = G_E\left(q_{zm_{12}} + 3q_{zm_{23}}\right) \tag{VII.387}$$

and the coefficients Γ_{K_4}, Γ_{K_5} *and* Γ_{K_6} *given by*

$$\Gamma_{K_4} = 0 \tag{VII.388}$$

$$\Gamma_{K_5} = 0 \tag{VII.389}$$

$$\Gamma_{K_6} = 0 \tag{VII.390}$$

where the coefficient G_E *is given by*

$$G_E = \frac{8\left(a_0^* - \tilde{a}_0\right) - \left(3\tilde{c}_{z1_7} + \tilde{c}_{z1_9} + \tilde{c}_{z2_8} + 3\tilde{c}_{z2_{10}}\right)}{\left(3q_{zm_{12}} + q_{zm_{23}}\right)^2 + \left(q_{zm_{13}} + q_{zm_{22}}\right)^2 + \left(q_{zm_{12}} + 3q_{zm_{23}}\right)^2} \tag{VII.391}$$

where the quadratic coefficients $q_{zm_{12}}$, $q_{zm_{13}}$, $q_{zm_{22}}$, *and* $q_{zm_{23}}$ *are the appropriate elements of the coefficient matrix* Q_{zm_1} *from the two dimensional quadratic normal form equation VII.194, and where* $\tilde{a}_0$ *from equation VII.326 and the coefficients* $\tilde{c}_{z1_7}$, $\tilde{c}_{z1_9}$, $\tilde{c}_{z2_8}$ *and* $\tilde{c}_{z2_{10}}$ *from equations VII.333 and VII.334 are evaluated for* $\Pi_L = 0$.

Proof. From the Linear Center Manifold Solution theorem in Chapter VI we have that $K_{\mu_1} = K_{z_1} = K_{z_2} = 0$ implies $\Pi_L = 0$ and vice-versa. Plugging $\Pi_L = 0$ into equation VII.339 yields a new equation to solve for Π_{Q_1}, which is

$$\Pi_{Q_1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 3q_{zm_{12}} + q_{zm_{23}} \\ q_{zm_{13}} + q_{zm_{22}} \\ q_{zm_{12}} + 3q_{zm_{23}} \end{bmatrix} = 8\left(a_0^* - \tilde{a}_0\right) - \left(3\tilde{c}_{z1_7} + \tilde{c}_{z1_9} + \tilde{c}_{z2_8} + 3\tilde{c}_{z2_{10}}\right) \tag{VII.392}$$

By inspection, we have that the corollary solutions for the coefficients $\Pi_{Q_{14}}$, $\Pi_{Q_{15}}$ and $\Pi_{Q_{16}}$ and G_E satisfy this equation, proving the first part of the corollary. To see that the components of Γ_K^T are zero, note that the matrix $\Gamma_z\left(\Pi_L\right)$ is zero when $\Pi_L = 0$, since $M_2\left(\Pi_L = 0\right) = 0$ in the first term, and since the second term is linearly dependent on Π_L. Then, since $\Gamma_z\left(\Pi_L\right) = 0$, we get $\Gamma_K^T = 0$ by equation VII.345, proving the corollary. ◁

7. Double Zero Bifurcations

Control systems in linear normal form with

$$F_z = \begin{bmatrix} 0 & \lambda_0 \\ 0 & 0 \end{bmatrix} \tag{VII.393}$$

($\lambda_0 \neq 0$) which implies

$$F_\mu = \begin{bmatrix} 0 \\ \beta \end{bmatrix} \tag{VII.394}$$

experience a linearly unstabilizable co-dimension one double zero bifurcation. Double zero bifurcations are discussed in Wiggins [Ref. 20] and are characterized by equations of the form

$$\dot{x}_1 = x_2 + O^{(3+)} \tag{VII.395}$$

$$\dot{x}_2 = \mu_1 + \mu_2 x_2 + x_1^2 \pm x_1 x_2 + O^{(3+)} \tag{VII.396}$$

which require both parameters, μ_1 and μ_2, for complete characterization. Thus, a co-dimension one double zero bifurcation can be considered a degenerate form of the complete co-dimension two case. Control of linearly unstabilizable double zero bifurcations is a topic which has not been addressed at this point, but remains a subject for future research.

8. Two Zeroes Bifurcations

Control systems in linear normal form with

$$F_z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{VII.397}$$

which implies

$$F_\mu = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{VII.398}$$

experience a linearly unstabilizable co-dimension one two zeroes bifurcation. However, two zeroes bifurcations require more than one parameter for complete characterization, so the co-dimension one case can be considered a degenerate form of the

complete case. For one approach to the solution of the two zeroes bifurcation, the reader is referred to Kang [Ref. 10].

VIII. EXAMPLES OF BIFURCATION CONTROL

In this chapter we will look at two examples of how the bifurcation control process really works. We will start with an example which is as simple as possible, to make it easy to follow. Then we will finish with the Moore-Greitzer model of rotating compressor stall in turbine engines, a problem which is the subject of active current research.

Throughout the preceding chapters we have developed a theoretical framework for controlling bifurcations. The theory was broadly general, and there were few opportunities to show how it worked. Now we want to put all the pieces together. Our first example is deliberately and artificially simple. Its virtue is that it is easy to follow, so that the details of the steps involved do not obscure the steps themselves.

A. EXAMPLE 1: CONTROLLING A SADDLE-NODE BIFURCATION IN A SIMPLE SYSTEM

Consider the following dynamic system:

$$\dot{\breve{\mu}} = 0 \tag{VIII.1}$$

$$\dot{\breve{z}} = \breve{\mu} + 4\breve{z}^2 + 5\breve{z}\breve{y} + \breve{y}^2 \tag{VIII.2}$$

$$\dot{\breve{y}} = \breve{u} \tag{VIII.3}$$

Although it looks rather complicated at first, it is actually quite simple. The system has only one parameter, $\breve{\mu}$; one linearly uncontrollable state, $\breve{z}$; one linearly controllable state, $\breve{y}$; and a single control input, $\breve{u}$. Also, as we will see, the system is already in quadratic normal form, and the coefficients have been specially chosen to make the math work out easily.

1. Analyze the System: Does an Undesirable Bifurcation Occur?

When is it necessary to go to the trouble of applying the techniques outlined in this dissertation to a given control system? If the system exhibits an undesirable bifurcation, then the techniques in this dissertation may be quite useful in finding a stabilizing controller. If not, then applying these techniques is probably a waste of effort. The key is to determine if an undesirable bifurcation occurs. This determination may be made in several ways. Anomalous experimental results may provide motivation for further analysis, simulations may indicate that a bifurcation occurs, or theoretical analysis may show the presence of a bifurcation in the system. Looking at the system in equations VIII.1 through VIII.3, we see that the $\breve{y}$ dynamics contain no non-linear terms and so are not a candidate for a bifurcation, and that the parameter $\breve{\mu}$ is a constant, which leaves only the uncontrollable state, $\breve{z}$. Does a bifurcation occur in the $\breve{z}$ dynamics? Looking at the $\breve{z}$ dynamics which would prevail if $\breve{y}$ were stabilized and held at zero, we have

$$\dot{\breve{z}} = \breve{\mu} + 4\breve{z}^2 \tag{VIII.4}$$

which clearly exhibits a saddle-node bifurcation as the parameter $\breve{\mu}$ changes from negative to positive. (When $\breve{\mu}$ is negative, the system has two equilibrium points, at $\breve{z}^* = \pm\frac{1}{2}\sqrt{-\breve{\mu}}$, where the notation $\breve{z}^*$ indicates an equilibrium point. When $\breve{\mu}$ is positive, the system does not have any equilibrium points, and $\breve{z}$ increases without limit. This qualitative change in the nature of a system's dynamics as a parameter is varied is what characterizes a bifurcation.) So, there is a distinct possibility that an undesirable bifurcation occurs in our system, and we would like to try and use the techniques outlined in Chapters II through VII to stabilize it.

2. Translate the Origin of Coordinates to the Desired Equilibrium Point at the Point of Bifurcation

We start by applying the techniques of Chapter II to find the point of bifurcation for the equilibrium point we are interested in, and then to translate the origin

of coordinates there. We need to do three things: find the equilibrium set at which the system can be trimmed using the control input $\breve{u}$; find the value of the parameter $\breve{\mu}$ at which the bifurcation occurs at the specific equilibrium point of interest (this is known as the point of bifurcation); and, translate the origin of coordinates to this point, renaming all of our variables appropriately.

a. Find the Trimmed Equilibrium Set

An equilibrium point is a point where the states of a system will not change if you put them there. The equilibrium set is all of the equilibrium points taken together. That means that we want to find all the points $\breve{\mu}$, $\breve{z} = \breve{z}^*$ and $\breve{y} = \breve{y}^*$ such that the derivatives of our states there are zero, i.e. $\dot{\mu} = 0$, $\dot{z} = 0$, and $\dot{y} = 0$. A trim control input is a constant control input which exactly maintains a system at an equilibrium point. For the equilibrium set, we may have a set of trim control inputs. That means that we want to find all the control inputs $\breve{u} = \breve{u}^*$ such that when the system is at an equilibrium point, it stays there. So, for our system, we find the equilibrium set and set of trim control inputs by setting all the derivatives equal to zero and solving for $\breve{\mu}$, $\breve{z} = \breve{z}^*$, $\breve{y} = \breve{y}^*$ and $\breve{u} = \breve{u}^*$. Looking at the first equation, we have $\dot{\mu} = 0$ always, which does not provide any new information. Looking at the third equation, we see that it gives us our trim control input, which turns out to be the set of trimmed control inputs. For $\dot{y} = 0$, we must take $\breve{u} = \breve{u}^* = 0$. It is the second equation which gives the equilibrium set. We are trying to find all the points where

$$\breve{\mu} + 4\breve{z}^{*2} + 5\breve{z}^*\breve{y}^* + \breve{y}^{*2} = 0 \qquad \text{(VIII.5)}$$

for the three independent variables $\breve{\mu}$, $\breve{z}^*$, and $\breve{y}^*$. Since equation VIII.5 is one equation in three unknowns, this is an underdetermined problem, and we will have two free variables, which we will pick as $\breve{\mu}$ and $\breve{y}^*$. Equation VIII.5 can be solved for $\breve{z}^*$ in terms of $\breve{\mu}$ and $\breve{y}^*$ to yield

$$\breve{z}^* = \frac{-5\breve{y}^* \pm \sqrt{9\breve{y}^{*2} - 16\breve{\mu}}}{8} \qquad \text{(VIII.6)}$$

Now, this is the general solution for the equilibrium set, which is rather complicated, even for this simple example. However, in most problems, we are not trying to find all the possible equilibrium points of a given system. Rather, we are only trying to find a single equilibrium point which is of engineering interest to us. Suddenly, the problem becomes very easy. Because we are trying to stabilize the system, and because $\breve{y}^*$ is a free variable, we can use external engineering considerations to pick a desired value of $\breve{y}^*$. The external engineering consideration for this example is that we desire simplicity, so we pick $\breve{y}^* = 0$. Now equation VIII.5 becomes

$$\breve{\mu} + 4\breve{z}^{*2} = 0 \tag{VIII.7}$$

which we solve easily as

$$\breve{z}^* = \pm\frac{1}{2}\sqrt{-\breve{\mu}} \tag{VIII.8}$$

So, our desired equilibrium point in terms of $\breve{\mu}$ and $\breve{u}$ is given by

$$\breve{z}^* = \pm\frac{1}{2}\sqrt{-\breve{\mu}} \tag{VIII.9}$$

$$\breve{y}^* = 0 \tag{VIII.10}$$

$$\breve{u}^* = 0 \tag{VIII.11}$$

b. Find the Point of Bifurcation

The bifurcation point is the value of the parameter at an equilibrium point where the dynamics of the system change qualitatively. For our simple system, this means the value of $\breve{\mu}$ at which the system changes from having two equilibrium points to having none. We will denote the value of the parameter at the bifurcation point by $\breve{\mu}^*$. Solving equation VIII.9, the point of bifurcation occurs at $\breve{\mu}^* = 0$, since all negative values of $\breve{\mu}$ produce two equilibrium points, and all positive values of $\breve{\mu}$ produce none. (If we needed to choose a value of $\breve{y}^* \neq 0$, we could solve equation VIII.6, and find that the point of bifurcation occurs at $\breve{\mu}^* = \frac{9\breve{y}^{*2}}{16}$.)

c. *Translate the Origin of Coordinates*

Now we need to translate the origin of coordinates to the equilibrium point of interest at the point of bifurcation. We use the coordinate translation

$$\mu = \breve{\mu} - \breve{\mu}^* \tag{VIII.12}$$

$$x_1 = \breve{z} - \breve{z}^* \tag{VIII.13}$$

$$x_2 = \breve{y} - \breve{y}^* \tag{VIII.14}$$

$$u = \breve{u} - \breve{u}^* \tag{VIII.15}$$

At the point of bifurcation, equation VIII.9 is given by

$$\breve{z}^* = \pm\frac{1}{2}\sqrt{-\breve{\mu}^*} \tag{VIII.16}$$

and since $\breve{\mu}^* = 0$, our coordinate translation is just

$$\mu = \breve{\mu} - 0 \tag{VIII.17}$$

$$x_1 = \breve{z} - 0 \tag{VIII.18}$$

$$x_2 = \breve{y} - 0 \tag{VIII.19}$$

$$u = \breve{u} - 0 \tag{VIII.20}$$

which yields our translated system as

$$\dot{\mu} = 0 \tag{VIII.21}$$

$$\dot{x}_1 = \mu + 4x_1^2 + 5x_1x_2 + x_2^2 \tag{VIII.22}$$

$$\dot{x}_2 = u \tag{VIII.23}$$

where the origin is the equilibrium point of interest at the point of bifurcation.

3. Put the System into Linear Normal Form

Next, we apply the techniques of Chapter III to put the system into linear normal form. Since the equilibrium point and point of bifurcation are now at the origin, we could put the system into linear normal form using the following steps:

- Taylor series expansion around the origin.
- Linear similarity transformation and linear state feedback.

However, because we have chosen our system as a special case, we note by inspection that our system is already in the form of a Taylor series expansion, and that the linear terms are already in Jordan-Brunovsky canonical form. From the Jordan-Brunovsky Canonical Form theorem in Chapter III, we have

$$\begin{bmatrix} \mu \\ x \end{bmatrix} = T \begin{bmatrix} \mu \\ \tilde{z} \\ \tilde{y} \end{bmatrix} \tag{VIII.24}$$

where we have anticipated the fact that we have no states $\tilde{w}$. By the same theorem, the linear similarity transformation T is given by

$$T = \begin{bmatrix} I & 0 \\ T_x T_\mu & T_x \end{bmatrix} \tag{VIII.25}$$

and the linear state feedback is given by

$$u = \tilde{u} - \alpha^T \mu - a^T \tilde{y} \tag{VIII.26}$$

By inspection of the problem we have

$$\begin{aligned} T_x &= I & \text{(VIII.27)} \\ T_\mu &= 0 & \text{(VIII.28)} \\ \alpha^T &= 0 & \text{(VIII.29)} \\ a^T &= 0 & \text{(VIII.30)} \end{aligned}$$

so our system in linear normal form is

$$\begin{bmatrix} \dot{\mu}_1 \\ \tilde{z}_1 \\ \tilde{y}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\mu}_1 \\ \tilde{z}_1 \\ \tilde{y}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tilde{u} + \begin{bmatrix} 0 \\ 4\tilde{z}_1^2 + 5\tilde{z}_1\tilde{y}_1 + \tilde{y}_1^2 \\ 0 \end{bmatrix} \qquad \text{(VIII.31)}$$

which gives

$$F_\mu = 1 \qquad \text{(VIII.32)}$$

$$F_z = 0 \qquad \text{(VIII.33)}$$

$$A = 0 \qquad \text{(VIII.34)}$$

$$B = 1 \qquad \text{(VIII.35)}$$

and where F_w does not exist in this system. The higher order terms are given by

$$\tilde{f}^{(2)}(\mu_1, \tilde{z}_1, \tilde{y}_1) = \begin{bmatrix} 0 \\ 4\tilde{z}_1^2 + 5\tilde{z}_1\tilde{y}_1 + \tilde{y}_1^2 \\ 0 \end{bmatrix} \qquad \text{(VIII.36)}$$

and

$$\tilde{g}^{(1)}(\mu_1, \tilde{z}_1, \tilde{y}_1) = \tilde{f}^{(3)}(\mu_1, \tilde{z}_1, \tilde{y}_1) = \tilde{g}^{(2)}(\mu_1, \tilde{z}_1, \tilde{y}_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(VIII.37)}$$

4. Put the System into Quadratic Normal Form

Examining equation VIII.31, the techniques of Chapter IV do not apply, since, although the system is not linearly unstable, the state $\tilde{z}_1$ is also not linearly stabilizable. So, we move on and apply the techniques of Chapter V to put the system into quadratic normal form. We apply a quadratic coordinate transformation of the form

$$\begin{bmatrix} \mu_1 \\ \tilde{z}_1 \\ \tilde{y}_1 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ z_1 \\ y_1 \end{bmatrix} + h^{(2)}(\mu_1, z_1, y_1) \qquad \text{(VIII.38)}$$

$$= \begin{bmatrix} \mu_1 \\ z_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ h_{21} & h_{22} & h_{23} & h_{24} & h_{25} & h_{26} \\ h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} \end{bmatrix} \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \\ \mu_1 y_1 \\ z_1 y_1 \\ y_1^2 \end{bmatrix}$$

where the coefficients h_{ij} which put the system into quadratic normal form are to be determined. The quadratic terms of our system in linear normal form are given by

$$\tilde{f}^{(2)}(\mu_1, z_1, y_1) = \begin{bmatrix} 0 \\ 4z_1^2 + 5z_1 y_1 + y_1^2 \\ 0 \end{bmatrix} \tag{VIII.39}$$

$$= \begin{bmatrix} \tilde{q}_{11} & \tilde{q}_{12} & \tilde{q}_{13} & \tilde{q}_{14} & \tilde{q}_{15} & \tilde{q}_{16} \\ \tilde{q}_{21} & \tilde{q}_{22} & \tilde{q}_{23} & \tilde{q}_{24} & \tilde{q}_{25} & \tilde{q}_{26} \\ \tilde{q}_{31} & \tilde{q}_{32} & \tilde{q}_{33} & \tilde{q}_{34} & \tilde{q}_{35} & \tilde{q}_{36} \end{bmatrix} \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \\ \mu_1 y_1 \\ z_1 y_1 \\ y_1^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \\ \mu_1 y_1 \\ z_1 y_1 \\ y_1^2 \end{bmatrix}$$

and

$$\tilde{g}^{(1)}(\mu_1, \tilde{z}_1, \tilde{y}_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{VIII.40}$$

$$= \begin{bmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} \\ \tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} \\ \tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} \end{bmatrix} \begin{bmatrix} \mu_1 \\ z_1 \\ y_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ z_1 \\ y_1 \end{bmatrix}$$

Using the five quadratic normal form theorems in Chapter V and the Poincare normal form for a saddle-node bifurcation from Appendix D, we can write our quadratic terms in normal form as

$$\check{f}^{(2)}(\mu_1, z_1, y_1) = \begin{bmatrix} 0 \\ \check{q}_{23} z_1^2 + \check{q}_{25} z_1 y_1 + \check{q}_{26} y_1^2 \\ 0 \end{bmatrix} \tag{VIII.41}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \check{q}_{23} & 0 & \check{q}_{25} & \check{q}_{26} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \\ \mu_1 y_1 \\ z_1 y_1 \\ y_1^2 \end{bmatrix}$$

and

$$\check{g}^{(1)}(\mu_1, \tilde{z}_1, \tilde{y}_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{VIII.42}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ z_1 \\ y_1 \end{bmatrix}$$

where the terms $\check{q}_{23}$, $\check{q}_{25}$ and $\check{q}_{26}$ are to be determined. Now, as we noted earlier, our simple system was carefully chosen to be in quadratic normal form from the

beginning. We could go through the steps outlined in the Unstacking Theorem in Chapter V and set up for a solution to $h^{(2)}(\mu_1, z_1, y_1)$ and $\check{q}_{23}$, $\check{q}_{25}$ and $\check{q}_{26}$. However, for this simple system we find that by inspection we can use the quadratic coordinate transformation

$$h^{(2)}(\mu_1, z_1, y_1) = 0 \tag{VIII.43}$$

and the quadratic control law

$$\tilde{u} = v \tag{VIII.44}$$

to solve for the quadratic normal form coefficients

$$\check{q}_{23} = 4 \tag{VIII.45}$$

$$\check{q}_{25} = 5 \tag{VIII.46}$$

$$\check{q}_{26} = 1 \tag{VIII.47}$$

The general quadratic normal form for a system with $\mu = [\mu_1] \in R^1$, $z = [z_1] \in R^1$, and $y = [y_1] \in R^1$ is

$$\dot{\mu}_1 = 0 \tag{VIII.48}$$

$$\begin{aligned} \dot{z}_1 &= F_\mu \mu_1 + Q_{z_{P_1}} \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \end{bmatrix} + Q_{z_{m_1}} \begin{bmatrix} \mu_1 y_1 \\ z_1 y_1 \end{bmatrix} + Q_{z_c} \begin{bmatrix} y_1^2 \end{bmatrix} \\ &+ f_z^{(3)}(\mu_1, z_1, y_1) + g_z^{(2)}(\mu_1, z_1, y_1) v + O^{(4+)} \end{aligned} \tag{VIII.49}$$

$$\dot{y}_1 = A y_1 + B v + Q_{y_c} \begin{bmatrix} y_1^2 \end{bmatrix} + O^{(3+)} \tag{VIII.50}$$

where we note that $A = 0$, $B = 1$ and $Q_{c_y} = 0$ for $y \in R^1$. Examining our system, we see that $F_\mu = 1$ and $F_z = 0$, which is indicative of a saddle-node bifurcation (which we already knew). $Q_{z_{P_1}}$ is the matrix of coefficients for the Poincare Normal Form quadratic terms, which has the following form for a saddle-node bifurcation: $Q_{z_{P_1}} = \begin{bmatrix} 0 & 0 & q_{z_{P_3}} \end{bmatrix}$. Now, examining our system term by term, we see that as expected our system is already in normal form, with the following coefficient matrices:

$$F_\mu = [1] \tag{VIII.51}$$

$$
\begin{aligned}
F_z &= [0] && \text{(VIII.52)}\\
Q_{z_{P_1}} &= \begin{bmatrix} 0 & 0 & q_{z_{P_3}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 \end{bmatrix} && \text{(VIII.53)}\\
Q_{z_{m_1}} &= \begin{bmatrix} q_{z_{m_1}} & q_{z_{m_2}} \end{bmatrix} = \begin{bmatrix} 0 & 5 \end{bmatrix} && \text{(VIII.54)}\\
Q_{z_c} &= \begin{bmatrix} q_{z_{c_1}} \end{bmatrix} = [1] && \text{(VIII.55)}\\
A &= [0] && \text{(VIII.56)}\\
B &= [1] && \text{(VIII.57)}\\
Q_{y_c} &= [0] && \text{(VIII.58)}
\end{aligned}
$$

Looking at the cubic order terms, and defining

$$\chi = \begin{bmatrix} \mu_1 \\ z_1 \\ y_1 \end{bmatrix} \tag{VIII.59}$$

$$F = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{VIII.60}$$

$$G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{VIII.61}$$

we have

$$\vec{f}^{(3)}(\chi) = \bar{f}^{(3)}(\chi) - \frac{\partial}{\partial \chi} h^{(2)}(\chi) \left(\tilde{f}^{(2)}(\chi) + F h^{(2)}(\chi) - \frac{\partial}{\partial \chi} h^{(2)}(\chi) F \chi \right) \tag{VIII.62}$$

$$\vec{g}^{(2)}(\chi) = \bar{g}^{(2)}(\chi) - \frac{\partial}{\partial \chi} h^{(2)}(\chi) \left(\tilde{g}^{(1)}(\chi) - \frac{\partial}{\partial \chi} h^{(2)}(\chi) G \right) \tag{VIII.63}$$

and

$$\bar{f}^{(3)}(\chi) + O^{(4+)} = \tilde{f}^{(3)}(\chi) + \left(\tilde{f}^{(2)}\left(\chi + h^{(2)}(\chi)\right) - \tilde{f}^{(2)}(\chi) \right) \tag{VIII.64}$$

$$\bar{g}^{(2)}(\chi) = \tilde{g}^{(2)}(\chi) + \tilde{g}^{(1)}\left(h^{(2)}(\chi)\right) \tag{VIII.65}$$

However, since our quadratic coordinate transformation is $h^{(2)}(\chi) = 0$, we have

$$\vec{f}^{(3)}(\chi) = \tilde{f}^{(3)}(\chi) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{VIII.66}$$

$$\vec{g}^{(2)}(\chi) = \tilde{g}^{(2)}(\chi) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{VIII.67}$$

Finally, we need to pick out the z_1 components of $\vec{f}^{(3)}(\chi)$ and $\vec{g}^{(2)}(\chi)$, which are

$$f_z^{(3)}(\mu_1, z_1, y_1) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \vec{f}^{(3)}(\chi) = \begin{bmatrix} 0 \end{bmatrix} \tag{VIII.68}$$

$$g_z^{(2)}(\mu_1, z_1, y_1) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \vec{g}^{(2)}(\chi) = \begin{bmatrix} 0 \end{bmatrix} \tag{VIII.69}$$

5. Stabilize the Linearly Controllable States

Now that our system is in quadratic normal form, we want to use the techniques in Chapter VI to cause our linearly controllable state y_1 to collapse onto the center manifold. We choose a state feedback gain of $K_{y_1} = -10$, which gives exponential stability for y_1 onto the center manifold with a decay time constant of 0.1.

6. Stabilize the Linearly Unstabilizable States

Now that we've stabilized the linearly controllable state y_1 onto the center manifold, we need to use the techniques of Chapter VII to stabilize or soften the bifurcation which occurs in z_1. Because the center manifold is one dimensional, we will be using linear state feedback of μ_1 and z_1 to attempt to exactly cancel the destabilizing quadratic term z_1^2, and we will be using quadratic state feedback of μ_1^2, $\mu_1 z_1$ and z_1^2 to produce a desired negative value for the coefficient of the equivalent z_1^3 term. Since we have a saddle-node bifurcation we are trying to stabilize, we use the Saddle-Node Linear Gains theorem from Chapter VII to calculate the gain K_{z_1} as

$$K_{z_1} = -K_{y_1} \Pi_{L_{1 z_1}} \tag{VIII.70}$$

where the coefficient $\Pi_{L_{1z_1}}$ is given by

$$\Pi_{L_{1z_1}} = \frac{-q_{zm_2} \pm \sqrt{\left(q_{zm_2}\right)^2 - 4q_{zc_1}q_{zP_3}}}{2q_{zc_1}} \tag{VIII.71}$$

with $q_{zP_3} = 4$, $q_{zm_2} = 5$ and $q_{zc_1} = 1$. Plugging in, we get $\Pi_{L_{1z_1}} = -1$ or -4 depending on whether the plus or minus sign is chosen. For simplicity, we choose $\Pi_{L_{1z_1}} = -1$, which gives $K_{z_1} = -10$. From the theorem, the gain K_{μ_1} can be chosen arbitrarily, and we choose $K_{\mu_1} = 0$. So, our linear gains are

$$K_{\mu_1} = 0 \tag{VIII.72}$$

$$K_{z_1} = -10 \tag{VIII.73}$$

$$K_{y_1} = -10 \tag{VIII.74}$$

To determine the quadratic gains, we use the Saddle-Node Quadratic Gains theorem from Chapter VII to calculate the gain $K_{z_1^2}$ as

$$K_{z_1^2} = -K_{y_1}\Pi_{Q_{1z_1^2}} + \Gamma_{K_3} \tag{VIII.75}$$

where the coefficient $\Pi_{Q_{1z_1^2}}$ is given by

$$\Pi_{Q_{1z_1^2}} = \frac{c^*_{cm_4} - \tilde{c}_{z_4}}{q_{zm_2} + 2q_{zc_1}\Pi_{L_{1z_1}}} \tag{VIII.76}$$

and where the coefficient Γ_{K_3} is given by

$$\Gamma_{K_3} = -\Pi_{L_{1z_1}}q_{cm_3} \tag{VIII.77}$$

Now, q_{cm_3} is the coefficient given by the formula

$$q_{cm_3} = q_{zP_3} + q_{zm_2}\Pi_{L_{1z_1}} + q_{zc_1}\left(\Pi_{L_{1z_1}}\right)^2 \tag{VIII.78}$$

and, as expected, our choice of linear gains forced it to zero. So, $\Gamma_{K_3} = 0$. This leaves the only two remaining unknowns as $c^*_{cm_4}$, which we are free to pick, and $\tilde{c}_{z_4}$, which we have to calculate from our system in quadratic normal form. Now $c^*_{cm_4}$ is the

coefficient of the z_1^3 term in our closed loop system after both linear and quadratic state feedback has been applied, and for cubic order non-linear stability we need $c_{cm_4}^* < 0$, so we pick $c_{cm_4}^* = -3$. The coefficient $\tilde{c}_{z_4}$ is from the z_1^3 term in the closed loop system after linear state feedback has been applied, but before quadratic state feedback has been applied. Since $f_z^{(3)}(\mu_1, z_1, y_1) = 0$ and $g_z^{(2)}(\mu_1, z_1, y_1) = 0$, we have $\tilde{c}_{z_4} = 0$. Plugging in, we get $\Pi_{Q_{1z_1^2}} = -1$, since $q_{z_{m_2}} = 5$ and $q_{z_{c_1}} = 1$, which gives $K_{z_1^2} = -10$. By assumption in the theorem, the quadratic state feedback gains $K_{\mu_1 y_1}$, $K_{z_1 y_1}$ and $K_{y_1^2}$ are all zero. The theorem also allows us to choose $K_{\mu_1^2}$ and $K_{\mu_1 z_1}$ arbitrarily, and we choose them to be zero. So, the quadratic state feedback gains are given by

$$K_{\mu_1^2} = 0 \tag{VIII.79}$$

$$K_{\mu_1 z_1} = 0 \tag{VIII.80}$$

$$K_{z_1^2} = -10 \tag{VIII.81}$$

$$K_{\mu_1 y_1} = 0 \tag{VIII.82}$$

$$K_{z_1 y_1} = 0 \tag{VIII.83}$$

$$K_{y_1^2} = 0 \tag{VIII.84}$$

7. Undo the Transformations

Now that we have the state feedback gains which stabilize the system in the transformed coordinate system, we need to reverse all the transformations to obtain the required control law in the original non-transformed system. This procedure was developed at the end of Chapter V, and resulted in the formula

$$\begin{aligned} u &= \left(K_\chi^T - \begin{bmatrix} \alpha^T & 0 & a^T \end{bmatrix}\right) T^{-1}\hat{\chi} \\ &- \left(G_\nu^T T^{-1}\hat{\chi}\right)\left(K_\chi^T T^{-1}\hat{\chi}\right) + \left(K_{\chi^{(2)}}^T - F_\nu^T - K_\chi^T H\right) \Upsilon \left(T^{-1}\right) \hat{\chi}^{(2)} \end{aligned} \tag{VIII.85}$$

where we have used

$$\hat{\chi} = \begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \end{bmatrix} \tag{VIII.86}$$

$$K_{\chi}^{T} = \begin{bmatrix} K_{\mu_1} & K_{z_1} & K_{y_1} \end{bmatrix} \tag{VIII.87}$$
$$= \begin{bmatrix} 0 & -10 & -10 \end{bmatrix}$$

$$K_{\chi^{(2)}}^{T} = \begin{bmatrix} K_{\mu_1^2} & K_{\mu_1 z_1} & K_{z_1^2} & K_{\mu_1 y_1} & K_{z_1 y_1} & K_{y_1^2} \end{bmatrix} \tag{VIII.88}$$
$$= \begin{bmatrix} 0 & 0 & -10 & 0 & 0 & 0 \end{bmatrix}$$

where we have from the linear coordinate transformation

$$\alpha^T = 0 \tag{VIII.89}$$
$$a^T = 0 \tag{VIII.90}$$
$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{VIII.91}$$

and from the quadratic coordinate transformation

$$G_{\nu}^{T} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \tag{VIII.92}$$
$$F_{\nu}^{T} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.93}$$
$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{VIII.94}$$

Finally, we have to calculate the matrix $\Upsilon(T^{-1})$, which is defined by the relation

$$\tilde{\chi}^{(2)} = \Upsilon\left(T^{-1}\right)\hat{\chi}^{(2)} \tag{VIII.95}$$

where

$$\tilde{\chi} = \begin{bmatrix} \mu_1 \\ \tilde{z}_1 \\ \tilde{y}_1 \end{bmatrix} \tag{VIII.96}$$

and where $\hat{\chi}$ was previously defined in equation VIII.86. We will get different values of $\Upsilon(T^{-1})$ depending on what order we choose to stack up the quadratic states, but

since $T^{-1} = I$ we can stack up $\tilde{\chi}^{(2)}$ the same as $\hat{\chi}^{(2)}$ and get

$$\Upsilon\left(T^{-1}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{VIII.97}$$

Putting all the pieces together yields a control law around the origin of

$$u = -10x_1 - 10x_2 - 10x_1^2 \tag{VIII.98}$$

Since our original system was at the origin, our fully developed control law is given as

$$\breve{u} = -10\breve{z} - 10\breve{y} - 10\breve{z}^2 \tag{VIII.99}$$

8. System Simulation

The last part of this example is simulating our results, to see if the gains we have calculated above really stabilize the saddle-node bifurcation. To restate, our system is

$$\dot{\mu} = 0 \tag{VIII.100}$$

$$\dot{z} = \mu + 4z^2 + 5zy + y^2 \tag{VIII.101}$$

$$\dot{y} = u \tag{VIII.102}$$

where we have dropped the "breve" notation for clarity ($\breve{z} \rightarrow z$, etc.), and where we will be applying the control law

$$\begin{aligned} u \;=\;& K_{\mu_1}\mu + K_{z_1}z + K_{y_1}y \\ +\;& K_{\mu_1^2}\mu^2 + K_{\mu_1 z_1}\mu z + K_{z_1^2}z^2 \\ +\;& K_{\mu_1 y_1}\mu y + K_{z_1 y_1}zy + K_{y_1^2}y^2 \end{aligned} \tag{VIII.103}$$

with the following value of gains:

$$K_{\mu_1} = 0 \tag{VIII.104}$$

$$K_{z_1} = -10 \tag{VIII.105}$$

$$K_{y_1} = -10 \tag{VIII.106}$$

$$K_{\mu_1^2} = 0 \tag{VIII.107}$$

$$K_{\mu_1 z_1} = 0 \tag{VIII.108}$$

$$K_{z_1^2} = -10 \tag{VIII.109}$$

$$K_{\mu_1 y_1} = 0 \tag{VIII.110}$$

$$K_{z_1 y_1} = 0 \tag{VIII.111}$$

$$K_{y_1^2} = 0 \tag{VIII.112}$$

We will run our simulation two ways: with state feedback, and without state feedback. In both cases, the initial condition for y will be zero, and the initial condition for z will be 0.1. The MATLAB program used to generate the simulation is presented in Appendix E. Results of the simulation are presented below in Figures 9 through 15. In Figures 9 through 11, the linearly controllable state y is stabilized with feedback ($K_{y_1} = -10$), but the bifurcation is not stabilized ($K_{z_1} = 0$, and $K_{z_1^2} = 0$), which we define as the "Feedback OFF" case. In Figures 12 through 15, feedback control is used to stabilize both the linearly controllable state y ($K_{y_1} = -10$), and the bifurcation ($K_{z_1} = -10$, and $K_{z_1^2} = -10$), which we define as the "Feedback ON" case.

In Figure 9, the value of the parameter is $\mu = -0.1$, and the simulation moves from an initial condition of $z = 0.1$ to a steady state value of approximately $z = -0.16$ at approximately $t = 5$. Theory predicts that two equilibrium points should exist in the uncontrolled dynamics, one at $z^* = -0.158$, which is stable, and one at $z^* = 0.158$, which is unstable. Since our initial condition is between the two equilibrium points, the theory predicts that the system should stabilize at $z = -0.158$, which is borne out by the simulation.

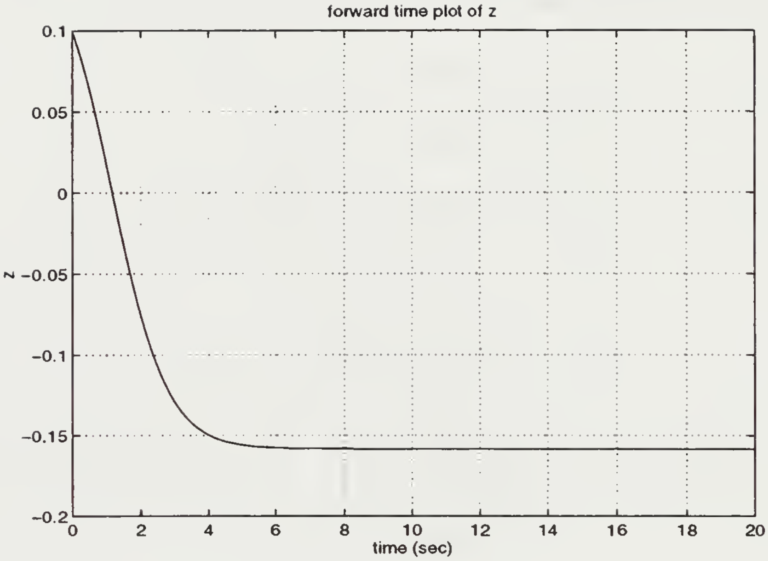

Figure 9. Simulation: z Dynamics for $\mu = -0.1$ (Feedback OFF)

In Figure 10, the value of the parameter is $\mu = 0$, and the simulation diverges from an initial condition of $z = 0.1$ apparently without limit, reaching a value of approximately $z = 0.5$ by $t = 2$. Theory predicts that this is the point of bifurcation,

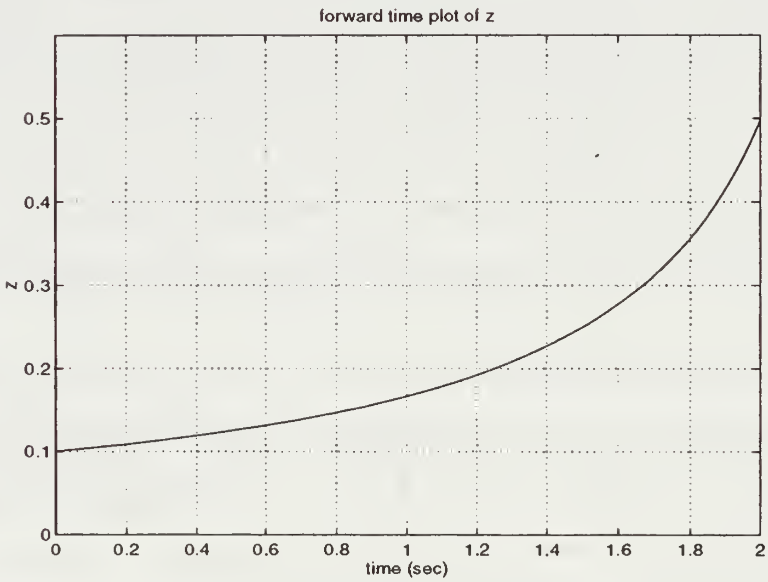

Figure 10. Simulation: z Dynamics for $\mu = 0$ (Feedback OFF)

and that only one equilibrium point should exist in the uncontrolled dynamics at

$z^* = 0$, which is "half-stable", that is, attractive for $z < 0$ and repulsive for $z > 0$. Since our initial condition is greater than zero, the theory predicts that z should diverge without limit, which is borne out by the simulation.

In Figure 11, the value of the parameter is $\mu = 0.1$, and the simulation diverges from an initial condition of $z = 0.1$ apparently without limit, reaching a value of approximately $z = 2.7$ by $t = 1.5$. Theory predicts that no equilibrium points

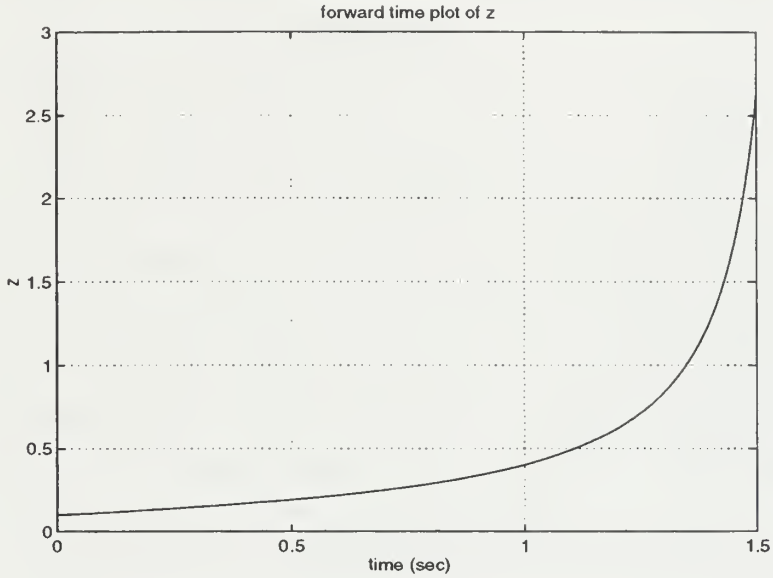

Figure 11. Simulation: z Dynamics for $\mu = 0.1$ (Feedback OFF)

should exist in the uncontrolled dynamics, and that z should diverge to positive values without limit, which is borne out by the simulation.

In Figures 12 through 15, feedback control is used to stabilize both the linearly controllable state y ($K_{y_1} = -10$), and the bifurcation ($K_{z_1} = -10$, and $K_{z_1^2} = -10$). Theory predicts that in this case the closed loop, linearly uncontrollable dynamics are given by the equation

$$\dot{z} = \mu - 3z^3 \qquad \text{(VIII.113)}$$

with a single equilibrium point at

$$z^* = \left(\frac{\mu}{3}\right)^{\frac{1}{3}} \qquad \text{(VIII.114)}$$

which is stable for all values of μ, and linearly stable for all values of $\mu \neq 0$, with an eigenvalue of

$$\lambda = -3\left(\frac{\mu}{3}\right)^{\frac{2}{3}} \tag{VIII.115}$$

In Figure 12, the value of the parameter is $\mu = -0.1$, and the simulation moves from an initial condition of $z = 0.1$ to a steady state value of approximately $z = -0.32$ at approximately $t = 8$. Theory predicts that for $\mu = -0.1$, the system

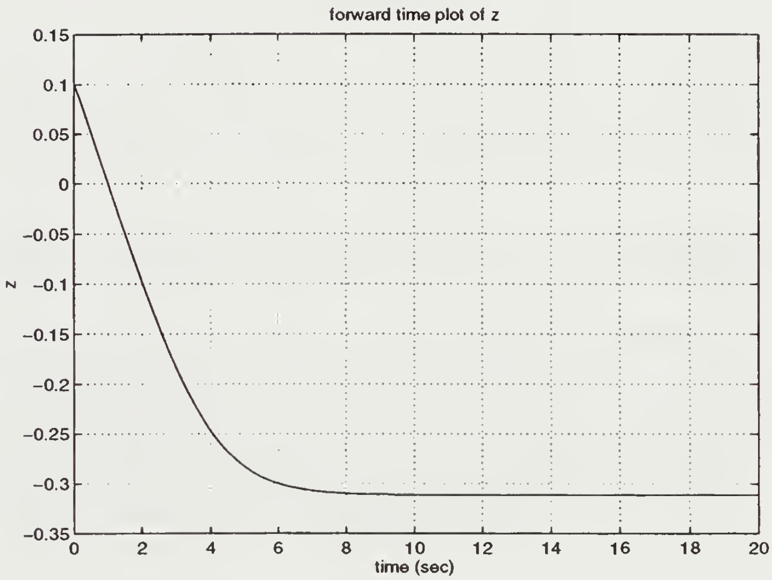

Figure 12. Simulation: z Dynamics for $\mu = -0.1$ (Feedback ON)

should be attracted to a linearly stable equilibrium point at $z^* = -0.322$, with an eigenvalue of $\lambda - 0.311$, which is borne out by the simulation.

In Figure 13, the value of the parameter is $\mu = 0$. From an initial condition of $z = 0.1$, and after an initial transient while y collapses to the center manifold (which lasts until approximately $t = 0.5$), z decreases slowly (algebraically), achieving a value of approximately $z = 0.07$ at $t = 20$, and apparently continuing to decay. Theory predicts that this is the point of bifurcation, and that the system should be attracted to a non-linearly stable equilibrium point at $z^* = 0$, with a zero eigenvalue, which is borne out by the simulation.

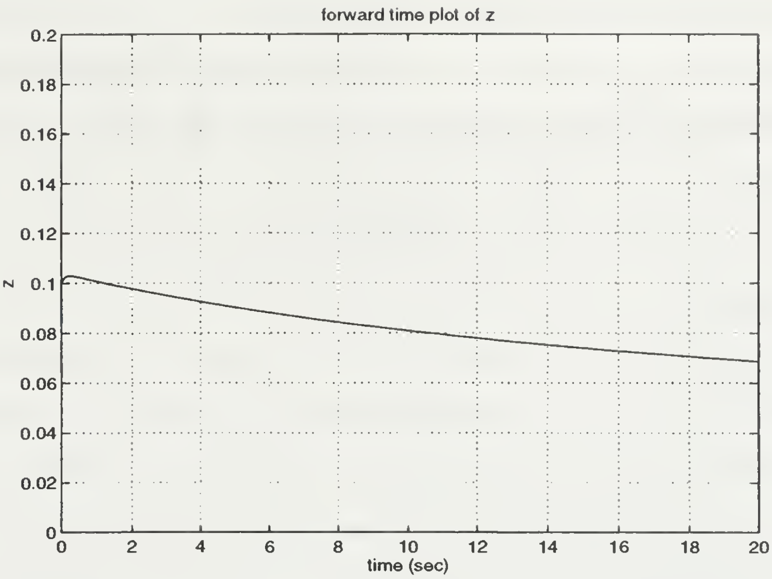

Figure 13. Simulation: z Dynamics for $\mu = 0$ (Feedback ON)

In Figure 14, the value of the parameter is $\mu = 0.1$, and the simulation moves from an initial condition of $z = 0.1$, to a steady state value of approximately $z = 0.33$ at approximately $t = 5$. Theory predicts that for $\mu = 0.1$, the system should be

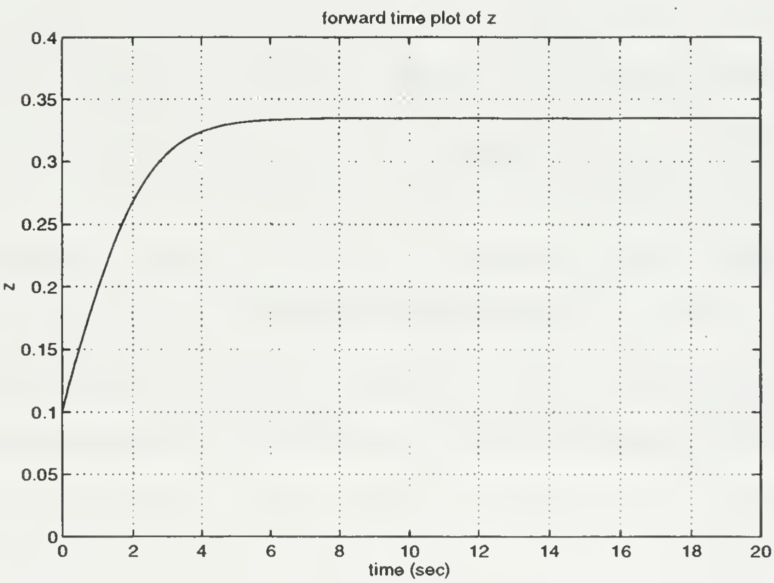

Figure 14. Simulation: z Dynamics for $\mu = 0.1$ (Feedback ON)

attracted to a linearly stable equilibrium point at $z^* = 0.322$, with an eigenvalue of

$\lambda = -0.311$, which is borne out by the simulation.

Finally, Figure 15 shows the collapse of the linearly controllable state y to the center manifold when stabilizing feedback is applied. The value of the parameter is

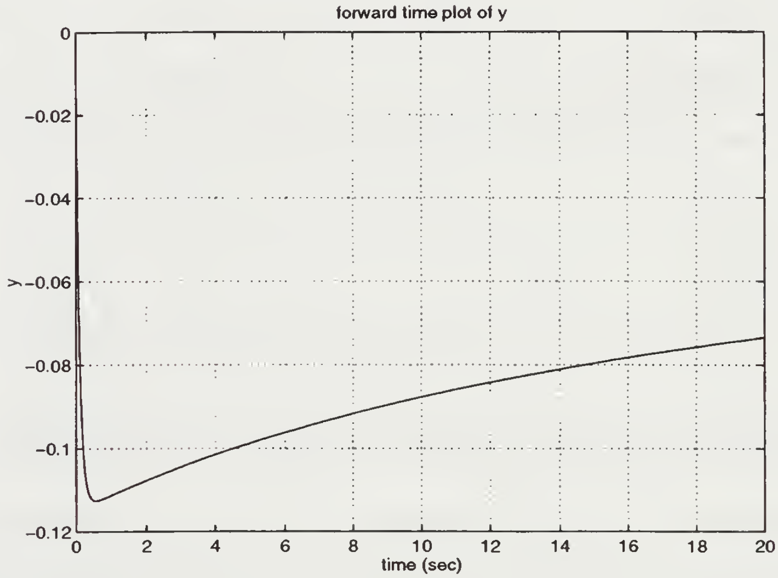

Figure 15. Simulation: y Dynamics for $\mu = 0$ (Feedback ON)

$\mu = 0$, and from an initial condition of $y = 0$, the simulation shows that y changes very rapidly (exponential decay), such that at $t = 0.5$, y has effectively collapsed to the center manifold at approximately $y = -0.11$. It then slowly moves back toward the origin in algebraic decay.

In conclusion, this simulation clearly shows the effect of applying the gains developed by the general method developed in Chapters II through VII to stabilize a saddle-node bifurcation. Figures 9 through 15 show an open loop unstable linearly uncontrollable system stabilized by non-linear control laws as developed in this dissertation.

B. EXAMPLE 2: THE MOORE GREITZER ENGINE COMPRESSOR MODEL

The Moore-Greitzer mathematical compressor model allows the rotating stall dynamics of a turbine engine compressor to be studied. The full Moore-Greitzer model consists of a non-linear partial differential equation and a non-linear ordinary differential equation. The partial differential equation can be approximated by using a Galerkin procedure, resulting in a set of non-linear ordinary differential equations for the pressure rise, Ψ, annulus averaged mass-flow, Φ, and spatial Fourier coefficients of the mass flow, A_n and B_n. In this example, we will consider the four state approximation, where only the first spatial harmonic coefficients of the Fourier expansion A_1 and B_1 are included. The four state Moore-Greitzer model of a turbine engine compressor is given by the following dynamic system

$$\dot{\mu} = 0 \tag{VIII.116}$$

$$\dot{A}_1 = \breve{\alpha}_1 \left[\left(\Psi_C' (\Phi) - \mu \right) A_1 - \frac{1}{b} B_1 + \frac{\Psi_C''' (\Phi)}{8} \left(A_1^3 + A_1 B_1^2 \right) \right] \tag{VIII.117}$$

$$\dot{B}_1 = \breve{\alpha}_1 \left[\frac{1}{b} A_1 + \left(\Psi_C' (\Phi) - \mu \right) B_1 + \frac{\Psi_C''' (\Phi)}{8} \left(A_1^2 B_1 + B_1^3 \right) \right] \tag{VIII.118}$$

$$\dot{\Phi} = \frac{1}{l_c} \left[\Psi_C (\Phi) - \Psi + \frac{\Psi_C'' (\Phi)}{4} \left(A_1^2 + B_1^2 \right) \right] \tag{VIII.119}$$

$$\dot{\Psi} = \frac{1}{4B^2 l_c} \left[\Phi - \Phi_T (\gamma, \Psi) \right] \tag{VIII.120}$$

where μ is a gas viscosity parameter which models viscous dissipation, and $\breve{\alpha}_1$, b, l_c and B are system constants. The system is quite complicated, and it gets worse, because both the compressor map, $\Psi_C(\Phi)$, and throttle map, $\Phi_T(\gamma, \Psi)$, are non-linear functions as well. Note that our control input will come from the throttle parameter, γ, by modulating the compressor bleed valve. We will begin our analysis of this system using the general case above, but later simplify the coefficients by plugging in more tractable numerical values.

1. Analyze the System: Does an Undesirable Bifurcation Occur?

The four state Moore-Greitzer model is the subject of significant current research, which has determined that numerous varieties of bifurcations occur at different compressor operating conditions. The bifurcation which we will be concerned with is the onset of rotating stall as the viscosity parameter, μ, takes on different values. This bifurcation is a Hopf bifurcation which occurs in the two states A_1 and B_1, and is characterized by the appearance or disappearance of a limit cycle in these two states. As we will see, the Hopf bifurcation can be either supercritical, where a stable limit cycle appears, or subcritical, where an unstable limit cycle disappears, forcing a previously stable equilibrium point into instability, and destabilizing the whole system. Our task in this example will be to calculate state feedback gains for our system which "flip" the catastrophic subcritical Hopf bifurcation into a more benign supercritical Hopf bifurcation.

2. Translate the Origin of Coordinates to the Desired Equilibrium Point at the Point of Bifurcation

As before, we need to do three things: first, find the desired equilibrium point in terms of the parameter μ, and the control input u, where we will define u as we progress; second, find the value of the parameter μ at which the bifurcation occurs at that equilibrium point (the point of bifurcation); and third, translate the origin of coordinates to this point, renaming all of our variables appropriately.

a. *Find the Equilibrium Points*

Since an equilibrium point is a point where the states of a system will not change if you put them there, we need to find a point where all the derivatives of our states are zero, i.e. $\dot{\mu} = 0$, $\dot{A}_1 = 0$, $\dot{B}_1 = 0$, $\dot{\Phi} = 0$, and $\dot{\Psi} = 0$. The first one is easy as always: In our system, $\dot{\mu} = 0$. Since we are trying to stabilize our system against rotating stall, we would like to find an equilibrium point where $A_1 = 0$ and

$B_1 = 0$ as well. Our equilibrium set is given by the equations

$$\left(\Psi_C'(\Phi^*) - \mu\right) A_1^* - \frac{1}{b} B_1^* + \frac{\Psi_C'''(\Phi^*)}{8}\left(A_1^{*3} + A_1^* B_1^{*2}\right) = 0 \quad \text{(VIII.121)}$$

$$\frac{1}{b} A_1^* + \left(\Psi_C'(\Phi^*) - \mu\right) B_1^* + \frac{\Psi_C'''(\Phi^*)}{8}\left(A_1^{*2} B_1^* + B_1^{*3}\right) = 0 \quad \text{(VIII.122)}$$

$$\Psi_C(\Phi^*) - \Psi^* + \frac{\Psi_C''(\Phi^*)}{4}\left(A_1^{*2} + B_1^{*2}\right) = 0 \quad \text{(VIII.123)}$$

$$\Phi^* - \Phi_T(\gamma^*, \Psi^*) = 0 \quad \text{(VIII.124)}$$

and we notice that if both $A_1^* = 0$ and $B_1^* = 0$, then the first two equations are automatically satisfied, and the third equation is greatly simplified. Now in Chapter II we learned that often one of the components of the equilibrium state vector would be a free variable, and that one of the components of the equilibrium equation would allow us to find the trim value of the control input, u^*. (Note that, aside from knowing that the control input u will somehow come from the throttle setting, γ, we still do not know what form it will have.) So, plugging in $A_1^* = 0$ and $B_1^* = 0$, and rearranging a little bit, the portion of the equilibrium set we are interested in is given by

$$A_1^* = 0 \quad \text{(VIII.125)}$$

$$B_1^* = 0 \quad \text{(VIII.126)}$$

$$\Psi^* = \Psi_C(\Phi^*) \quad \text{(VIII.127)}$$

$$\Phi^* - \Phi_T(\gamma^*, \Psi_C(\Phi^*)) = 0 \quad \text{(VIII.128)}$$

where we have plugged in for Ψ^* in the fourth equation to make it obvious that Φ^* is the component of the equilibrium state vector which we have chosen as a free variable. Now it is time to bite the bullet and look at the equations for the compressor map, $\Psi_C(\Phi)$, and throttle map, $\Phi_T(\gamma, \Psi)$. The compressor map is given by

$$\Psi_C(\Phi) = \Psi_{C_0} + \Psi_{C_1}\Phi + \Psi_{C_2}\Phi^2 + \Psi_{C_3}\Phi^3 + O^{(4+)} \quad \text{(VIII.129)}$$

with

$$\Psi_C'(\Phi) = \Psi_{C_1} + 2\Psi_{C_2}\Phi + 3\Psi_{C_3}\Phi^2 + O^{(3+)} \quad \text{(VIII.130)}$$

$$\Psi_C''(\Phi) = 2\Psi_{C_2} + 6\Psi_{C_3}\Phi + O^{(2+)} \tag{VIII.131}$$

$$\Psi_C'''(\Phi) = 6\Psi_{C_3} + O^{(1+)} \tag{VIII.132}$$

where the coefficients are typically obtained by a third order curve fit to empirical measurements, and the throttle map is given by

$$\Phi_T(\gamma, \Psi) = \gamma\sqrt{\Psi} \tag{VIII.133}$$

where γ, the throttle parameter, is proportional to the effective area that the air pressurized by the compressor has to flow through. So, we see that if we can open a bleed valve on the high pressure side of the compressor, we can change the effective area of the flow, and affect γ.

Now we are ready to take the final step. By plugging our compressor map and throttle map formulas into our equilibrium set equations, we find our equilibrium throttle setting, γ^*, and our equilibrium pressure rise, Ψ^*, as functions of our equilibrium mass flow rate, Φ^*, which is a free variable. (In actuality of course, the throttle and other engine controls are what are varied as needed to achieve the desired mass flow, which we are treating here as a free variable.) So, with Φ^* as our free parameter, our equilibrium set of interest is

$$A_1^* = 0 \tag{VIII.134}$$

$$B_1^* = 0 \tag{VIII.135}$$

$$\Psi^* = \Psi_C(\Phi^*) = \Psi_{C_0} + \Psi_{C_1}\Phi^* + \Psi_{C_2}\Phi^{*2} + \Psi_{C_3}\Phi^{*3} + O^{(4+)} \tag{VIII.136}$$

and our trim control input is

$$\gamma^* = \frac{\Phi^*}{\sqrt{\Psi_C(\Phi^*)}} = \frac{\Phi^*}{\sqrt{\Psi_{C_0} + \Psi_{C_1}\Phi^* + \Psi_{C_2}\Phi^{*2} + \Psi_{C_3}\Phi^{*3} + O^{(4+)}}} \tag{VIII.137}$$

How do we choose Φ^*? We use engineering judgement. We want our compressor to produce a pressure rise, so let's pick the point of maximum pressure rise. For example, for the specific compressor map we will be considering later

$$\Psi_C(\Phi) = -1 + 3\Phi - \Phi^3 + O^{(4+)} \tag{VIII.138}$$

the maximum equilibrium pressure rise is $\Psi^*_{\max} = 1$, which occurs at an equilibrium flow rate of $\Phi^* = 1$, as shown in Figure 16. So, having chosen our equilibrium point,

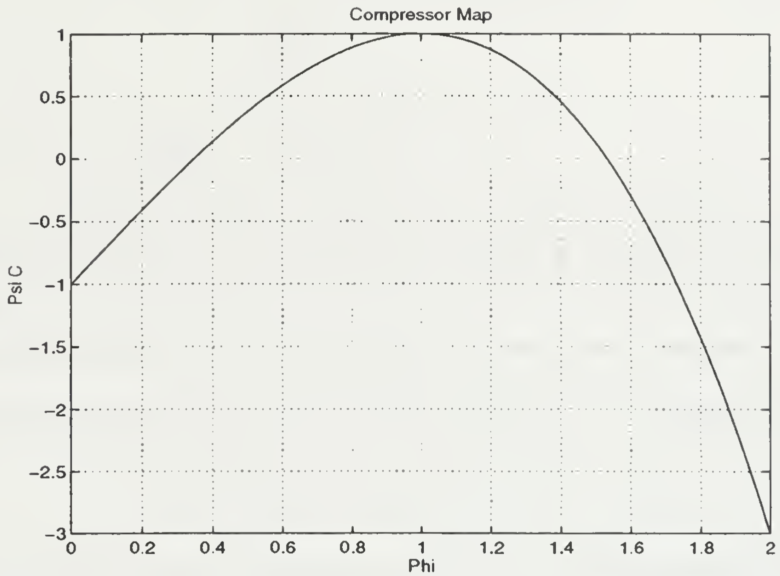

Figure 16. Compressor Map: $\Psi_C(\Phi) = -1 + 3\Phi - \Phi^3$

we can move on to the next step.

b. Find the Point of Bifurcation

A bifurcation point is the value of a parameter at an equilibrium point where the dynamics of the system change qualitatively. The qualitative change could be any of a number of possible conditions, including a change in the number of equilibrium points, a change in the stability of an equilibrium point, the creation or destruction of limit cycles, etc. A quick check reveals that our equilibrium set of interest, equations VIII.134, VIII.135 and VIII.136, do not depend on the parameter μ, and so changes in the value of the parameter cannot force a change in the number of equilibrium points. To check for changes in the stability of an equilibrium point, we need to linearize our system around the equilibrium point of interest and check for values of the parameter which cause the real part of any eigenvalue or eigenvalues to become zero. Linearizing by taking the Jacobian of equations VIII.117, VIII.118, VIII.119 and VIII.120 evaluated at the equilibrium point $A_1^* = 0$, $B_1^* = 0$, $\Psi^* =$

$\Psi_C(\Phi^*)$ and $\Phi = \Phi^*$, we get

$$J = \begin{bmatrix} \breve{\alpha}_1\left(\Psi_C'(\Phi^*) - \mu\right) & -\frac{\breve{\alpha}_1}{b} & 0 & 0 \\ \frac{\breve{\alpha}_1}{b} & \breve{\alpha}_1\left(\Psi_C'(\Phi^*) - \mu\right) & 0 & 0 \\ 0 & 0 & \frac{\Psi_C'(\Phi^*)}{l_c} & \frac{-1}{l_c} \\ 0 & 0 & \frac{1}{4B^2 l_c} & \frac{1}{8B^2 l_c}\gamma(\Phi^*)\left(\Psi_C(\Phi^*)\right)^{-\frac{1}{2}} \end{bmatrix} \tag{VIII.139}$$

where J is the Jacobian of our system evaluated at the equilibrium point. Since the matrix J is block diagonal, we can evaluate the eigenvalues of the entire matrix by evaluating the eigenvalues of the individual blocks, and only the upper left block depends on the parameter μ. The characteristic equation of the upper left block is

$$\left(\lambda - \breve{\alpha}_1\left(\Psi_C'(\Phi^*) - \mu\right)\right)^2 + \left(\frac{\breve{\alpha}_1}{b}\right)^2 = 0 \tag{VIII.140}$$

which shows that the eigenvalues λ are purely imaginary (and therefore have zero real parts) when $\mu = \Psi_C'(\Phi^*)$. So, we pick

$$\mu^* = \Psi_C'(\Phi^*) \tag{VIII.141}$$

as the point of bifurcation.

c. *Translate the Origin of Coordinates*

Now we need to translate the origin of coordinates of our system to the equilibrium point at the point of bifurcation. We start by restating the equations of our control system, (VIII.116) through (VIII.120), in fully written out form as

$$\dot{\mu}_1 = 0 \tag{VIII.142}$$

$$\begin{aligned} \dot{A}_1 &= \breve{\alpha}_1\left(\Psi_{C_1} + 2\Psi_{C_2}\Phi + 3\Psi_{C_3}\Phi^2 + O^{(3+)} - \mu\right)A_1 \\ &- \frac{\breve{\alpha}_1}{b}B_1 + \breve{\alpha}_1\frac{\left(6\Psi_{C_3} + O^{(1+)}\right)}{8}\left(A_1^3 + A_1B_1^2\right) \end{aligned} \tag{VIII.143}$$

$$\begin{aligned} \dot{B}_1 &= \frac{\breve{\alpha}_1}{b}A_1 + \breve{\alpha}_1\left(\Psi_{C_1} + 2\Psi_{C_2}\Phi + 3\Psi_{C_3}\Phi^2 + O^{(3+)} - \mu\right)B_1 \\ &+ \breve{\alpha}_1\frac{\left(6\Psi_{C_3} + O^{(1+)}\right)}{8}\left(A_1^2B_1 + B_1^3\right) \end{aligned} \tag{VIII.144}$$

$$\dot{\Phi} = \frac{1}{l_c}\left(\Psi_{C_0} + \Psi_{C_1}\Phi + \Psi_{C_2}\Phi^2 + \Psi_{C_3}\Phi^3 + O^{(4+)} - \Psi\right) + \frac{\left(2\Psi_{C_2} + 6\Psi_{C_3}\Phi + O^{(2+)}\right)}{4l_c}\left(A_1^2 + B_1^2\right) \tag{VIII.145}$$

$$\dot{\Psi} = \frac{1}{4B^2 l_c}\left[\Phi - \gamma\sqrt{\Psi}\right] \tag{VIII.146}$$

where we have plugged in equations VIII.129 through VIII.133 for the compressor map and the throttle map. Defining our new translated variables in terms of the old variables gives

$$\mu_1 = \mu - \mu^* = \mu - \Psi_C'(\Phi^*) \tag{VIII.147}$$

$$x_1 = A_1 - A_1^* = A_1 \tag{VIII.148}$$

$$x_2 = B_1 - B_1^* = B_1 \tag{VIII.149}$$

$$x_3 = \Phi - \Phi^* \tag{VIII.150}$$

$$x_4 = \Psi - \Psi^* = \Psi - \Psi_C(\Phi^*) \tag{VIII.151}$$

$$u = \gamma - \gamma^* = \gamma - \frac{\Phi^*}{\sqrt{\Psi_C(\Phi^*)}} \tag{VIII.152}$$

which we can plug into equations VIII.142 through VIII.146 after rearranging to get

$$\dot{\mu}_1 = 0 \tag{VIII.153}$$

$$\dot{x}_1 = -\breve{\alpha}_1\mu_1 x_1 - \frac{\breve{\alpha}_1}{b}x_2 + \breve{\alpha}_1\Psi_C''(\Phi^*)\,x_1x_3 + \frac{\breve{\alpha}_1}{8}\Psi_C'''(\Phi^*)\left(x_1^3 + x_1x_2^2 + 4x_1x_3^2\right) + O^{(4+)} \tag{VIII.154}$$

$$\dot{x}_2 = \frac{\breve{\alpha}_1}{b}x_1 - \breve{\alpha}_1\mu_1 x_2 + \breve{\alpha}_1\Psi_C''(\Phi^*)\,x_2x_3 + \frac{\breve{\alpha}_1}{8}\Psi_C'''(\Phi^*)\left(x_1^2x_2 + x_2^3 + 4x_2x_3^2\right) + O^{(4+)} \tag{VIII.155}$$

$$\dot{x}_3 = \frac{1}{l_c}\Psi_C'(\Phi^*)\,x_3 - \frac{1}{l_c}x_4 + \frac{\Psi_C''(\Phi^*)}{4l_c}\left(x_1^2 + x_2^2 + 2x_3^2\right) + \frac{\Psi_C'''(\Phi^*)}{12l_c}\left(3x_1^2x_3 + 3x_2^2x_3 + 2x_3^3\right) + O^{(4+)} \tag{VIII.156}$$

$$\dot{x}_4 = \frac{1}{4B^2 l_c} x_3 + \frac{\Phi^*}{4B^2 l_c}\left(1 - \sqrt{1 + \frac{x_4}{\Psi_C(\Phi^*)}}\right) \qquad \text{(VIII.157)}$$
$$- \frac{1}{4B^2 l_c}\sqrt{\Psi_C(\Phi^*)}\left(\sqrt{1 + \frac{x_4}{\Psi_C(\Phi^*)}}\right) u$$

where we have used the definitions

$$\Psi_C(\Phi^*) = \Psi_{C_0} + \Psi_{C_1}\Phi^* + \Psi_{C_2}\Phi^{*2} + \Psi_{C_3}\Phi^{*3} + O^{(4+)} \qquad \text{(VIII.158)}$$

$$\Psi_C'(\Phi^*) = \Psi_{C_1} + 2\Psi_{C_2}\Phi^* + 3\Psi_{C_3}\Phi^{*2} + O^{(3+)} \qquad \text{(VIII.159)}$$

$$\Psi_C''(\Phi^*) = 2\Psi_{C_2} + 6\Psi_{C_3}\Phi^* + O^{(2+)} \qquad \text{(VIII.160)}$$

$$\Psi_C'''(\Phi^*) = 6\Psi_{C_3} + O^{(1+)} \qquad \text{(VIII.161)}$$

Equations VIII.153 through VIII.157 are our equations of motion around the equilibrium point of interest at the point of bifurcation.

3. Put the System into Linear Normal Form

Next, we apply the techniques of Chapter III to put the system given by equations VIII.153 through VIII.157 into linear normal form. We do this by expanding our system in a multivariable Taylor series around the origin, including all terms through third order and performing a linear similarity transformation with applied linear state feedback, to put the linear terms of the system into Jordan-Brunovsky canonical form. We start with the Taylor series expansion.

a. Taylor Series Expansion Around the Origin

Looking at the system given by equations VIII.153 through VIII.157, we see that all of the equations are already in Taylor series expansion form except for equation VIII.157. Equation VIII.157 has a Taylor series expansion of the form

$$\dot{x}_4 = \frac{1}{4B^2 l_c}\left(x_3 + \Phi^*\left(-\frac{1}{2}\frac{x_4}{\Psi_C(\Phi^*)} + \frac{1}{8}\frac{x_4^2}{(\Psi_C(\Phi^*))^2} - \frac{1}{16}\frac{x_4^3}{(\Psi_C(\Phi^*))^3} + O^{(4+)}\right)\right) \qquad \text{(VIII.162)}$$
$$- \frac{1}{4B^2 l_c}\sqrt{\Psi_C(\Phi^*)}\left(1 + \frac{1}{2}\frac{x_4}{\Psi_C(\Phi^*)} - \frac{1}{8}\frac{x_4^2}{(\Psi_C(\Phi^*))^2} + \frac{1}{16}\frac{x_4^3}{(\Psi_C(\Phi^*))^3} + O^{(4+)}\right) u$$

We can further break out our Taylor series expansion by putting our set of system equations into vector matrix form in the linear terms, as follows

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\breve{\alpha}_1}{b} & 0 & 0 \\ 0 & \frac{\breve{\alpha}_1}{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\Psi_C'(\Phi^*)}{l_c} & -\frac{1}{l_c} \\ 0 & 0 & 0 & \frac{1}{4B^2 l_c} & -\frac{\Phi^*}{8B^2 l_c \Psi_C(\Phi^*)} \end{bmatrix} \begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{VIII.163}$$
$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{4B^2 l_c}\sqrt{\Psi_C(\Phi^*)} \end{bmatrix} u + f^{(2+)}(x,\mu_1) + g^{(1+)}(x,\mu_1)\, u$$

where the notation $f^{(2+)}(x,\mu_1)$ and $g^{(1+)}(x,\mu_1)$ denotes functions of of order 2 and higher and 1 and higher, respectively. These functions are given by

$$f^{(2+)}(x,\mu_1) = \begin{bmatrix} 0 \\ \breve{\alpha}_1\left(-\mu_1 x_1 + \Psi_C''(\Phi^*)\,x_1 x_3 + \frac{1}{8}\Psi_C'''(\Phi^*)\left(x_1^3 + x_1 x_2^2 + 4x_1 x_3^2\right)\right) \\ \breve{\alpha}_1\left(-\mu_1 x_2 + \Psi_C''(\Phi^*)\,x_2 x_3 + \frac{1}{8}\Psi_C'''(\Phi^*)\left(x_1^2 x_2 + x_2^3 + 4x_2 x_3^2\right)\right) \\ \frac{1}{l_c}\left(\frac{\Psi_C''(\Phi^*)}{4}\left(x_1^2 + x_2^2 + 2x_3^2\right) + \frac{\Psi_C'''(\Phi^*)}{12}\left(3x_1^2 x_3 + 3x_2^2 x_3 + 2x_3^3\right)\right) \\ \frac{1}{4B^2 l_c}\Phi^*\left(\frac{1}{8}\frac{x_4^2}{(\Psi_C(\Phi^*))^2} - \frac{1}{16}\frac{x_4^3}{(\Psi_C(\Phi^*))^3}\right) \end{bmatrix} + O^{(4+)} \tag{VIII.164}$$

and

$$g^{(1+)}(x,\mu_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{4B^2 l_c}\sqrt{\Psi_C(\Phi^*)}\left(\frac{1}{2}\frac{x_4}{\Psi_C(\Phi^*)} - \frac{1}{8}\frac{x_4^2}{(\Psi_C(\Phi^*))^2} + \frac{1}{16}\frac{x_4^3}{(\Psi_C(\Phi^*))^3} + O^{(4+)}\right) \end{bmatrix} \tag{VIII.165}$$

This completes our Taylor series expansion.

b. *Linear Similarity Transformation and Linear State Feedback*

Now lets look at the linear terms of our system. Counting the parameter μ_1, we have five states, so we have to produce a similarity transformation for a 5×5 matrix. Since in general finding a similarity transformation requires finding the eigenvalues and eigenvectors of a matrix, we rapidly run into trouble trying to complete this process symbolically. Instead, we will proceed numerically, picking values for the system constants $\breve{\alpha}_1$, b, l_c, and B, and picking a specific compressor map function, $\Psi_C(\Phi^*)$. (Note that in this particular case, since our 5×5 matrix is block diagonal, we could have proceeded symbolically, since we only had to transform two 2×2 matrices. However, since we are trying to illustrate a general procedure, we proceed numerically since that is the general case.) For our system, we pick

$$\breve{\alpha}_1 = 1 \tag{VIII.166}$$

$$b = 1 \tag{VIII.167}$$

$$l_c = \frac{1}{2} \tag{VIII.168}$$

$$B = \frac{1}{2} \tag{VIII.169}$$

$$\Psi_C(\Phi) = -1 + 3\Phi - \Phi^3 \tag{VIII.170}$$

which gives

$$\Phi^* = \Phi_{\max\Psi} = 1 \tag{VIII.171}$$

$$\Psi_C(\Phi^*) = 1 \tag{VIII.172}$$

$$\Psi_C'(\Phi^*) = 0 \tag{VIII.173}$$

$$\Psi_C''(\Phi^*) = -6 \tag{VIII.174}$$

$$\Psi_C'''(\Phi^*) = -6 \tag{VIII.175}$$

Now we can plug these values into our system equations and get

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} u + f^{(2+)}(x, \mu_1) + g^{(1+)}(x, \mu_1) u \tag{VIII.176}$$

with

$$f^{(2+)}(x, \mu_1) = \begin{bmatrix} 0 \\ -\mu_1 x_1 - 6x_1 x_3 - \frac{3}{4}\left(x_1^3 + x_1 x_2^2 + 4x_1 x_3^2\right) \\ -\mu_1 x_2 - 6x_2 x_3 - \frac{3}{4}\left(x_1^2 x_2 + x_2^3 + 4x_2 x_3^2\right) \\ -3\left(x_1^2 + x_2^2 + 2x_3^2\right) - \left(3x_1^2 x_3 + 3x_2^2 x_3 + 2x_3^3\right) \\ \frac{1}{4}x_4^2 - \frac{1}{8}x_4^3 \end{bmatrix} + O^{(4+)} \tag{VIII.177}$$

and

$$g^{(1+)}(x, \mu_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -x_4 + \frac{1}{4}x_4^2 - \frac{1}{8}x_4^3 \end{bmatrix} + O^{(4+)} \tag{VIII.178}$$

So, using the Jordan-Brunovsky Canonical Form theorem from Chapter III, we can apply a similarity transformation

$$\begin{bmatrix} \mu_1 \\ x \end{bmatrix} = T \begin{bmatrix} \mu_1 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} \tag{VIII.179}$$

and a linear state feedback

$$u = \tilde{u} - \alpha^T \mu_1 - a^T \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} \tag{VIII.180}$$

where the matrices T and T^{-1} are given by

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \tag{VIII.181}$$

and

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} \tag{VIII.182}$$

(Note that, in general, finding the transformation matrix T may be an involved process.) So, making the transformation, we have

$$\begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = T \begin{bmatrix} \mu_1 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ 4\tilde{y}_1 \\ -2\tilde{y}_2 \end{bmatrix} \tag{VIII.183}$$

and

$$\begin{bmatrix} \mu_1 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = T^{-1} \begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \\ \frac{1}{4}x_3 \\ -\frac{1}{2}x_4 \end{bmatrix} \tag{VIII.184}$$

which we plug into equation VIII.176 to get

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \\ \dot{\tilde{y}}_1 \\ \dot{\tilde{y}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4 & -1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u + O^{(2+)} \tag{VIII.185}$$

This gives

$$\alpha^T = [0] \tag{VIII.186}$$

$$a^T = \begin{bmatrix} -4 & -1 \end{bmatrix} \tag{VIII.187}$$

which lets us calculate the linear state feedback as

$$u = \tilde{u} + 4\tilde{y}_1 + \tilde{y}_2 \tag{VIII.188}$$

From Chapter III we have

$$\begin{aligned} \tilde{f}^{(2)}(\mu_1, \tilde{z}, \tilde{y}) &= T^{-1}\left[f^{(2)}\left(T \begin{bmatrix} \mu_1 \\ \tilde{z} \\ \tilde{y} \end{bmatrix} \right) \right] \\ &- T^{-1}\left[g^{(1)}\left(T \begin{bmatrix} \mu_1 \\ \tilde{z} \\ \tilde{y} \end{bmatrix} \right) \right] \left(\alpha^T \mu_1 + a^T \tilde{y} \right) \end{aligned} \tag{VIII.189}$$

$$\begin{aligned} \tilde{f}^{(3)}(\mu_1, \tilde{z}, \tilde{y}) &= T^{-1}\left[f^{(3)}\left(T \begin{bmatrix} \mu_1 \\ \tilde{z} \\ \tilde{y} \end{bmatrix} \right) \right] \\ &- T^{-1}\left[g^{(2)}\left(T \begin{bmatrix} \mu_1 \\ \tilde{z} \\ \tilde{y} \end{bmatrix} \right) \right] \left(\alpha^T \mu_1 + a^T \tilde{y} \right) \end{aligned} \tag{VIII.190}$$

and

$$\tilde{g}^{(1)}(\mu_1, \tilde{z}, \tilde{y}) = T^{-1}\left[g^{(1)}\left(T\begin{bmatrix}\mu_1 \\ \tilde{z} \\ \tilde{y}\end{bmatrix}\right)\right] \tag{VIII.191}$$

$$\tilde{g}^{(2)}(\mu_1, \tilde{z}, \tilde{y}) = T^{-1}\left[g^{(2)}\left(T\begin{bmatrix}\mu_1 \\ \tilde{z} \\ \tilde{y}\end{bmatrix}\right)\right] \tag{VIII.192}$$

So, plugging in VIII.183, VIII.184, VIII.177 and VIII.178 we get

$$\tilde{f}^{(2)}(\mu_1, \tilde{z}_1, \tilde{z}_2, \tilde{y}_1, \tilde{y}_2) = \begin{bmatrix} 0 \\ -\mu_1\tilde{z}_1 - 24\tilde{z}_1\tilde{y}_1 \\ -\mu_1\tilde{z}_2 - 24\tilde{z}_2\tilde{y}_1 \\ -\frac{3}{4}\tilde{z}_1^2 - \frac{3}{4}\tilde{z}_2^2 - 24\tilde{y}_1^2 \\ -4\tilde{y}_1\tilde{y}_2 - \frac{3}{2}\tilde{y}_2^2 \end{bmatrix} \tag{VIII.193}$$

$$\tilde{f}^{(3)}(\mu_1, \tilde{z}_1, \tilde{z}_2, \tilde{y}_1, \tilde{y}_2) = \begin{bmatrix} 0 \\ -\frac{3}{4}\tilde{z}_1^3 - \frac{3}{4}\tilde{z}_1\tilde{z}_2^2 - 48\tilde{z}_1\tilde{y}_1^2 \\ -\frac{3}{4}\tilde{z}_1^2\tilde{z}_2 - \frac{3}{4}\tilde{z}_2^3 - 48\tilde{z}_2\tilde{y}_1^2 \\ -3\tilde{z}_1^2\tilde{y}_1 - 3\tilde{z}_2^2\tilde{y}_1 - 32\tilde{y}_1^3 \\ -2\tilde{y}_1\tilde{y}_2^2 - \tilde{y}_2^3 \end{bmatrix} \tag{VIII.194}$$

and

$$\tilde{g}^{(1)}(\mu_1, \tilde{z}_1, \tilde{z}_2, \tilde{y}_1, \tilde{y}_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\tilde{y}_2 \end{bmatrix} \tag{VIII.195}$$

$$\tilde{g}^{(2)}(\mu_1, \tilde{z}_1, \tilde{z}_2, \tilde{y}_1, \tilde{y}_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2}\tilde{y}_2^2 \end{bmatrix} \tag{VIII.196}$$

which produces our system in linear normal form,

$$\begin{aligned} \begin{bmatrix} \dot{\mu}_1 \\ \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \\ \dot{\tilde{y}}_1 \\ \dot{\tilde{y}}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tilde{u} \\ &+ \tilde{f}^{(2)}(\mu_1, \tilde{z}_1, \tilde{z}_2, \tilde{y}_1, \tilde{y}_2) + \tilde{g}^{(1)}(\mu_1, \tilde{z}_1, \tilde{z}_2, \tilde{y}_1, \tilde{y}_2)\,\tilde{u} \\ &+ \tilde{f}^{(3)}(\mu_1, \tilde{z}_1, \tilde{z}_2, \tilde{y}_1, \tilde{y}_2) + \tilde{g}^{(2)}(\mu_1, \tilde{z}_1, \tilde{z}_2, \tilde{y}_1, \tilde{y}_2)\,\tilde{u} + O^{(4+)} \end{aligned} \tag{VIII.197}$$

where the terms $\tilde{f}^{(2)}(\mu_1, \tilde{z}_1, \tilde{z}_2, \tilde{y}_1, \tilde{y}_2)$, $\tilde{f}^{(3)}(\mu_1, \tilde{z}_1, \tilde{z}_2, \tilde{y}_1, \tilde{y}_2)$, $\tilde{g}^{(1)}(\mu_1, \tilde{z}_1, \tilde{z}_2, \tilde{y}_1, \tilde{y}_2)$ and $\tilde{g}^{(2)}(\mu_1, \tilde{z}_1, \tilde{z}_2, \tilde{y}_1, \tilde{y}_2)$ are defined above.

4. Put the System into Quadratic Normal Form

Examining equation VIII.197, the techniques of Chapter IV do not apply, since, although the system is not linearly unstable, the states $\tilde{z}_1$ and $\tilde{z}_2$ are not linearly stabilizable. So, we move on and apply the techniques of Chapter V to put the system into quadratic normal form. We apply a quadratic coordinate transformation of the form

$$\begin{bmatrix} \mu \\ \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} \mu \\ z_1 \\ z_2 \\ y_1 \\ y_2 \end{bmatrix} + h^{(2)}(\mu_1, z_1, z_2, y_1, y_2) \tag{VIII.198}$$

$$
= \begin{bmatrix} \mu \\ z_1 \\ z_2 \\ y_1 \\ y_2 \end{bmatrix} + H \begin{bmatrix} \mu_1^2 \\ \mu_1 z \\ z^{(2)} \\ \mu_1 y_1 \\ z y_1 \\ \mu_1 y_2 \\ z y_2 \\ y^{(2)} \end{bmatrix}
$$

where $H \in R^{5\times15}$ is a matrix of coefficients of the block form

$$
H = \begin{bmatrix} H_{\sigma u} & H_{\sigma m} & H_{\sigma c} \\ H_{yu} & H_{ym} & H_{yc} \end{bmatrix} \tag{VIII.199}
$$

with the blocks given by

$$
H_{\sigma u} = \begin{bmatrix} h_{1_1} & h_{1_2} & h_{1_3} & h_{1_4} & h_{1_5} & h_{1_6} \\ h_{2_1} & h_{2_2} & h_{2_3} & h_{2_4} & h_{2_5} & h_{2_6} \\ h_{3_1} & h_{3_2} & h_{3_3} & h_{3_4} & h_{3_5} & h_{3_6} \end{bmatrix} \tag{VIII.200}
$$

$$
H_{\sigma m} = \begin{bmatrix} h_{1_7} & h_{1_8} & h_{1_9} & h_{1_{10}} & h_{1_{11}} & h_{1_{12}} \\ h_{2_7} & h_{2_8} & h_{2_9} & h_{2_{10}} & h_{2_{11}} & h_{2_{12}} \\ h_{3_7} & h_{3_8} & h_{3_9} & h_{3_{10}} & h_{3_{11}} & h_{3_{12}} \end{bmatrix} \tag{VIII.201}
$$

$$
H_{\sigma c} = \begin{bmatrix} h_{1_{13}} & h_{1_{14}} & h_{1_{15}} \\ h_{2_{13}} & h_{2_{14}} & h_{2_{15}} \\ h_{3_{13}} & h_{3_{14}} & h_{3_{15}} \end{bmatrix} \tag{VIII.202}
$$

$$
H_{yu} = \begin{bmatrix} h_{4_1} & h_{4_2} & h_{4_3} & h_{4_4} & h_{4_5} & h_{4_6} \\ h_{5_1} & h_{5_2} & h_{5_3} & h_{5_4} & h_{5_5} & h_{5_6} \end{bmatrix} \tag{VIII.203}
$$

$$
H_{ym} = \begin{bmatrix} h_{4_7} & h_{4_8} & h_{4_9} & h_{4_{10}} & h_{4_{11}} & h_{4_{12}} \\ h_{5_7} & h_{5_8} & h_{5_9} & h_{5_{10}} & h_{5_{11}} & h_{5_{12}} \end{bmatrix} \tag{VIII.204}
$$

$$H_{yc} = \begin{bmatrix} h_{4_{13}} & h_{4_{14}} & h_{4_{15}} \\ h_{5_{13}} & h_{5_{14}} & h_{5_{15}} \end{bmatrix} \qquad \text{(VIII.205)}$$

We wish to determine the coefficients h_{i_j} which will transform our system into quadratic normal form. The quadratic terms of our system in linear normal form are given by

$$\begin{aligned} \tilde{f}^{(2)}(\mu_1, z_1, z_2, y_1, y_2) &= \begin{bmatrix} 0 \\ -\mu_1 z_1 - 24 z_1 y_1 \\ -\mu_1 z_2 - 24 z_2 y_1 \\ -\frac{3}{4} z_1^2 - \frac{3}{4} z_2^2 - 24 y_1^2 \\ -4 y_1 y_2 - \frac{3}{2} y_2^2 \end{bmatrix} \qquad \text{(VIII.206)} \\ &= \tilde{Q} \begin{bmatrix} \mu_1^2 \\ \mu_1 z \\ z^{(2)} \\ \mu_1 y_1 \\ z y_1 \\ \mu_1 y_2 \\ z y_2 \\ y^{(2)} \end{bmatrix} \end{aligned}$$

where $\tilde{Q} \in R^{5 \times 15}$ is a matrix of coefficients given by

$$\tilde{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -24 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{4} & 0 & -\frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & -24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & -\frac{3}{2} \end{bmatrix} \qquad \text{(VIII.207)}$$

which can be broken down into block form as

$$\tilde{Q} = \begin{bmatrix} \tilde{Q}_{\sigma u} & \tilde{Q}_{\sigma m} & \tilde{Q}_{\sigma c} \\ \tilde{Q}_{yu} & \tilde{Q}_{ym} & \tilde{Q}_{yc} \end{bmatrix} \qquad \text{(VIII.208)}$$

with

$$\tilde{Q}_{\sigma u} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.209}$$

$$\tilde{Q}_{\sigma m} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -24 & 0 & 0 & 0 & 0 \\ 0 & 0 & -24 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.210}$$

$$\tilde{Q}_{\sigma c} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{VIII.211}$$

$$\tilde{Q}_{yu} = \begin{bmatrix} 0 & 0 & 0 & -\frac{3}{4} & 0 & -\frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.212}$$

$$\tilde{Q}_{ym} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.213}$$

$$\tilde{Q}_{yc} = \begin{bmatrix} -24 & 0 & 0 \\ 0 & -4 & -\frac{3}{2} \end{bmatrix} \tag{VIII.214}$$

Also

$$\begin{aligned} \tilde{g}^{(1)}(\mu_1, z_1, z_2, y_1, y_2) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -y_2 \end{bmatrix} \qquad \text{(VIII.215)} \\ &= \tilde{G} \begin{bmatrix} \mu \\ z_1 \\ z_2 \\ y_1 \\ y_2 \end{bmatrix} \end{aligned}$$

where $\tilde{G} \in R^{5\times 5}$ is a matrix of coefficients given by

$$\tilde{G} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{VIII.216}$$

which can be broken down in block form as

$$\tilde{G} = \begin{bmatrix} \tilde{G}_{\sigma u} & \tilde{G}_{\sigma c} \\ \tilde{G}_{yu} & \tilde{G}_{yc} \end{bmatrix} \tag{VIII.217}$$

with

$$\tilde{G}_{\sigma u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{VIII.218}$$

$$\tilde{G}_{\sigma c} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{VIII.219}$$

$$\tilde{G}_{yu} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{VIII.220}$$

$$\tilde{G}_{yc} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{VIII.221}$$

Using the five quadratic normal form theorems in Chapter V and the Poincare normal form for a Hopf bifurcation from Appendix D, we can write our quadratic terms in normal form as

$$\check{f}^{(2)}\left(\mu_1, z_1, z_2, y_1, y_2\right) \tag{VIII.222}$$

$$= \begin{bmatrix} 0 \\ \alpha_1\mu_1 z_1 - \omega_1\mu_1 z_2 + \check{q}_{27}\mu_1 y_1 + \check{q}_{28} z_1 y_1 + \check{q}_{29} z_2 y_1 + \check{q}_{213} y_1^2 + \check{q}_{215} y_2^2 \\ \omega_1\mu_1 z_1 + \alpha_1\mu_1 z_2 + \check{q}_{37}\mu_1 y_1 + \check{q}_{38} z_1 y_1 + \check{q}_{39} z_2 y_1 + \check{q}_{313} y_1^2 + \check{q}_{315} y_2^2 \\ 0 \\ \check{f}_\nu^{(2)}(\mu_1, z_1, z_2, y_1, y_2) \end{bmatrix}$$

$$= \check{Q} \begin{bmatrix} \mu_1^2 \\ \mu_1 z \\ z^{(2)} \\ \mu_1 y_1 \\ z y_1 \\ \mu_1 y_2 \\ z y_2 \\ y^{(2)} \end{bmatrix}$$

where $\check{Q} \in R^{5\times 15}$ is a matrix of coefficients given by

$$\check{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & -\omega_1 & 0 & 0 & 0 & \check{q}_{27} & \check{q}_{28} & \check{q}_{29} & 0 & 0 & 0 & \check{q}_{213} & 0 & \check{q}_{215} \\ 0 & \omega_1 & \alpha_1 & 0 & 0 & 0 & \check{q}_{37} & \check{q}_{38} & \check{q}_{39} & 0 & 0 & 0 & \check{q}_{313} & 0 & \check{q}_{315} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \check{q}_{\nu 1} & \check{q}_{\nu 2} & \check{q}_{\nu 3} & \check{q}_{\nu 4} & \check{q}_{\nu 5} & \check{q}_{\nu 6} & \check{q}_{\nu 7} & \check{q}_{\nu 8} & \check{q}_{\nu 9} & \check{q}_{\nu 10} & \check{q}_{\nu 11} & \check{q}_{\nu 12} & \check{q}_{\nu 13} & \check{q}_{\nu 14} & \check{q}_{\nu 15} \end{bmatrix} \tag{VIII.223}$$

with α_1, ω_1 and $\check{q}_{ij}$ coefficients to be determined, and which can be broken down into block form as

$$\check{Q} = \begin{bmatrix} \check{Q}_{\sigma u} & \check{Q}_{\sigma m} & \check{Q}_{\sigma c} \\ \check{Q}_{yu} & \check{Q}_{ym} & \check{Q}_{yc} \end{bmatrix} \tag{VIII.224}$$

where

$$\check{Q}_{\sigma u} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & -\omega_1 & 0 & 0 & 0 \\ 0 & \omega_1 & \alpha_1 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.225}$$

$$\check{Q}_{\sigma m} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \check{q}_{27} & \check{q}_{28} & \check{q}_{29} & 0 & 0 & 0 \\ \check{q}_{37} & \check{q}_{38} & \check{q}_{39} & 0 & 0 & 0 \end{bmatrix} \tag{VIII.226}$$

$$\check{Q}_{\sigma c} = \begin{bmatrix} 0 & 0 & 0 \\ \check{q}_{213} & 0 & \check{q}_{215} \\ \check{q}_{313} & 0 & \check{q}_{315} \end{bmatrix} \tag{VIII.227}$$

$$\check{Q}_{yu} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \check{q}_{\nu 1} & \check{q}_{\nu 2} & \check{q}_{\nu 3} & \check{q}_{\nu 4} & \check{q}_{\nu 5} & \check{q}_{\nu 6} \end{bmatrix} \tag{VIII.228}$$

$$\check{Q}_{ym} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \check{q}_{\nu 7} & \check{q}_{\nu 8} & \check{q}_{\nu 9} & \check{q}_{\nu 10} & \check{q}_{\nu 11} & \check{q}_{\nu 12} \end{bmatrix} \tag{VIII.229}$$

$$\check{Q}_{yc} = \begin{bmatrix} 0 & 0 & 0 \\ \check{q}_{\nu 13} & \check{q}_{\nu 14} & \check{q}_{\nu 15} \end{bmatrix} \tag{VIII.230}$$

Also

$$\begin{aligned} \check{g}^{(1)}\left(\mu_1, z_1, z_2, y_1, y_2\right) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \check{g}_{\nu}^{(1)}\left(\mu_1, z_1, z_2, y_1, y_2\right) \end{bmatrix} \\ &= \check{G} \begin{bmatrix} \mu \\ z_1 \\ z_2 \\ y_1 \\ y_2 \end{bmatrix} \end{aligned} \tag{VIII.231}$$

where $\check{G} \in R^{5\times 5}$ is a matrix of coefficients given by

$$\check{G} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \check{g}_{\nu 1} & \check{g}_{\nu 2} & \check{g}_{\nu 3} & \check{g}_{\nu 4} & \check{g}_{\nu 5} \end{bmatrix} \tag{VIII.232}$$

which can be broken down in block form as

$$\check{G} = \begin{bmatrix} \check{G}_{\sigma u} & \check{G}_{\sigma c} \\ \check{G}_{yu} & \check{G}_{yc} \end{bmatrix} \tag{VIII.233}$$

with

$$\check{G}_{\sigma u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{VIII.234}$$

$$\check{G}_{\sigma c} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{VIII.235}$$

$$\check{G}_{yu} = \begin{bmatrix} 0 & 0 & 0 \\ \check{g}_{\nu 1} & \check{g}_{\nu 2} & \check{g}_{\nu 3} \end{bmatrix} \tag{VIII.236}$$

$$\check{G}_{yc} = \begin{bmatrix} 0 & 0 \\ \check{g}_{\nu 4} & \check{g}_{\nu 5} \end{bmatrix} \tag{VIII.237}$$

We now have block decompositions for the matrices H, $\tilde{Q}$, $\tilde{G}$, $\check{Q}$ and $\check{G}$. From the Separation Principle theorem and "Constraints on H from g(x)" lemma in Chapter V, we can determine the matrices H, $\check{Q}$ and $\check{G}$ given the matrices $\tilde{Q}$ and $\tilde{G}$, by solving the block matrix equations

$$H_{yc}D_c - AH_{yc} = \tilde{Q}_{yc} - \check{Q}_{yc} \tag{VIII.238}$$

$$H_{ym}D_m - AH_{ym} = \tilde{Q}_{ym} - \check{Q}_{ym} \tag{VIII.239}$$

$$H_{yu}D_\sigma - AH_{yu} = \tilde{Q}_{yu} - \check{Q}_{yu} \tag{VIII.240}$$

$$H_{\sigma c}D_c - F_\sigma H_{\sigma c} = \tilde{Q}_{\sigma c} - \check{Q}_{\sigma c} \tag{VIII.241}$$

$$H_{\sigma m}D_m - F_\sigma H_{\sigma m} = \tilde{Q}_{\sigma m} - \check{Q}_{\sigma m} \tag{VIII.242}$$

$$H_{\sigma u}D_\sigma - F_\sigma H_{\sigma u} = \tilde{Q}_{\sigma u} - \check{Q}_{\sigma u} \tag{VIII.243}$$

subject to the constraints of the block matrix equations

$$H_{yc}D_{B_y} = \tilde{G}_{yc} - \check{G}_{yc} \tag{VIII.244}$$

$$H_{ym}D_{B_\sigma} = \tilde{G}_{yu} - \check{G}_{yu} \tag{VIII.245}$$

$$H_{\sigma c}D_{B_y} = \tilde{G}_{\sigma c} - \check{G}_{\sigma c} \tag{VIII.246}$$

$$H_{\sigma m}D_{B_\sigma} = \tilde{G}_{\sigma u} - \check{G}_{\sigma u} \tag{VIII.247}$$

For this problem, the matrices A, F_σ, D_c, D_m, D_σ, D_{B_y} and D_{B_σ} are given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{VIII.248}$$

$$F_\sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{VIII.249}$$

$$D_c = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{VIII.250}$$

$$D_m = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{VIII.251}$$

$$D_\sigma = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix} \tag{VIII.252}$$

$$D_{B_y} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \tag{VIII.253}$$

$$D_{B_\sigma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{VIII.254}$$

where we have used the definitions given in Appendix A and plugged in $\omega_0 = 1$. Now, using the Unstacking Theorem in Chapter V and the method of Appendix B, we can rewrite equations VIII.238 and VIII.244 in block form as

$$\begin{bmatrix} D_c^T & -I \\ 0 & D_c^T \end{bmatrix} \begin{bmatrix} H_{yc_1}^T \\ H_{yc_2}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{yc_1}^T \\ \tilde{Q}_{yc_2}^T \end{bmatrix} - \begin{bmatrix} 0 \\ \check{Q}_{yc_2}^T \end{bmatrix} \tag{VIII.255}$$

$$\begin{bmatrix} D_{B_y}^T & 0 \\ 0 & D_{B_y}^T \end{bmatrix} \begin{bmatrix} H_{yc_1}^T \\ H_{yc_2}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{yc_1}^T \\ \tilde{G}_{yc_2}^T \end{bmatrix} - \begin{bmatrix} 0 \\ \check{G}_{yc_2}^T \end{bmatrix} \tag{VIII.256}$$

which has the solution

$$H_{yc} = \begin{bmatrix} 0 & 0 & 0 \\ 24 & 0 & 0 \end{bmatrix} \tag{VIII.257}$$

$$\check{Q}_{yc} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -52 & -\frac{3}{2} \end{bmatrix} \tag{VIII.258}$$

$$\check{G}_{yc} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{VIII.259}$$

where the problem was underdetermined, and where we chose the coefficient $h_{4_{13}}$ as a free variable, which we set to zero, and where the bottom row of $\check{Q}_{yc}$ and $\check{G}_{yc}$ will be removed by state feedback after the coordinate transformation is complete. In like fashion, we can rewrite equations VIII.239 and VIII.245 in block form as

$$\begin{bmatrix} D_m^T & -I \\ 0 & D_m^T \end{bmatrix} \begin{bmatrix} H_{ym_1}^T \\ H_{ym_2}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{ym_1}^T \\ \tilde{Q}_{ym_2}^T \end{bmatrix} - \begin{bmatrix} 0 \\ \check{Q}_{ym_2}^T \end{bmatrix} \tag{VIII.260}$$

$$\begin{bmatrix} D_{B_\sigma}^T & 0 \\ 0 & D_{B_\sigma}^T \end{bmatrix} \begin{bmatrix} H_{ym_1}^T \\ H_{ym_2}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{yu_1}^T \\ \tilde{G}_{yu_2}^T \end{bmatrix} - \begin{bmatrix} 0 \\ \check{G}_{yu_2}^T \end{bmatrix} \tag{VIII.261}$$

which has the trivial solution

$$H_{ym} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.262}$$

$$\check{Q}_{ym} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.263}$$

$$\check{G}_{yu} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{VIII.264}$$

since both $\tilde{Q}_{ym} = 0$ and $\tilde{G}_{yu} = 0$. Continuing, we rewrite equation VIII.240 in block form as

$$\begin{bmatrix} D_\sigma^T & -I \\ 0 & D_\sigma^T \end{bmatrix} \begin{bmatrix} H_{yu_1}^T \\ H_{yu_2}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{yu_1}^T \\ \tilde{Q}_{yu_2}^T \end{bmatrix} - \begin{bmatrix} 0 \\ \check{Q}_{yu_2}^T \end{bmatrix} \tag{VIII.265}$$

which has the solution

$$H_{yu} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{3}{4} \end{bmatrix} \tag{VIII.266}$$

$$\check{Q}_{yu} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.267}$$

where we note that $\check{Q}_{yu} = 0$ because of a coincidental cancellation of non-zero terms, rather than for any fundamental reason. Continuing, we rewrite equations VIII.241 and VIII.246 in block form as

$$\begin{bmatrix} D_c^T & 0 & 0 \\ 0 & D_c^T & I \\ 0 & -I & D_c^T \end{bmatrix} \begin{bmatrix} H_{\sigma c_1}^T \\ H_{\sigma c_2}^T \\ H_{\sigma c_3}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{\sigma c_1}^T \\ \tilde{Q}_{\sigma c_2}^T \\ \tilde{Q}_{\sigma c_3}^T \end{bmatrix} - \begin{bmatrix} \check{Q}_{\sigma c_1}^T \\ \check{Q}_{\sigma c_2}^T \\ \check{Q}_{\sigma c_3}^T \end{bmatrix} \tag{VIII.268}$$

$$\begin{bmatrix} D_{B_y}^T & 0 & 0 \\ 0 & D_{B_y}^T & 0 \\ 0 & 0 & D_{B_y}^T \end{bmatrix} \begin{bmatrix} H_{\sigma c_1}^T \\ H_{\sigma c_2}^T \\ H_{\sigma c_3}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{\sigma c_1}^T \\ \tilde{G}_{\sigma c_2}^T \\ \tilde{G}_{\sigma c_3}^T \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{VIII.269}$$

which has the trivial solution

$$H_{\sigma c} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{VIII.270}$$

$$\check{Q}_{\sigma c} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{VIII.271}$$

$$\check{G}_{\sigma c} \doteq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{VIII.272}$$

since both $\check{Q}_{\sigma c} = 0$ and $\tilde{G}_{\sigma c} = 0$. Continuing, we rewrite equations VIII.242 and VIII.247 in block form as

$$\begin{bmatrix} D_m^T & 0 & 0 \\ 0 & D_m^T & I \\ 0 & -I & D_m^T \end{bmatrix} \begin{bmatrix} H_{\sigma m_1}^T \\ H_{\sigma m_2}^T \\ H_{\sigma m_3}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{\sigma m_1}^T \\ \tilde{Q}_{\sigma m_2}^T \\ \tilde{Q}_{\sigma m_3}^T \end{bmatrix} - \begin{bmatrix} \check{Q}_{\sigma m_1}^T \\ \check{Q}_{\sigma m_2}^T \\ \check{Q}_{\sigma m_3}^T \end{bmatrix} \tag{VIII.273}$$

$$\begin{bmatrix} D_{B_\sigma}^T & 0 & 0 \\ 0 & D_{B_\sigma}^T & 0 \\ 0 & 0 & D_{B_\sigma}^T \end{bmatrix} \begin{bmatrix} H_{\sigma m_1}^T \\ H_{\sigma m_2}^T \\ H_{\sigma m_3}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{\sigma u_1}^T \\ \tilde{G}_{\sigma u_2}^T \\ \tilde{G}_{\sigma u_3}^T \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{VIII.274}$$

which has the solution

$$H_{\sigma m} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.275}$$

$$\check{Q}_{\sigma m} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -24 & 0 & 0 & 0 & 0 \\ 0 & 0 & -24 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.276}$$

$$\check{G}_{\sigma u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{VIII.277}$$

where $\check{G}_{\sigma u} = 0$ by the "Constraints on H from g(x)" lemma in Chapter V, and where $H_{\sigma m} = 0$ by the fact that $\tilde{Q}_{\sigma m}$ was already in the proper normal form required for $\check{Q}_{\sigma m}$. Finally, we rewrite equation VIII.243 in block form as

$$\begin{bmatrix} D_\sigma^T & 0 & 0 \\ 0 & D_\sigma^T & I \\ 0 & -I & D_\sigma^T \end{bmatrix} \begin{bmatrix} H_{\sigma u_1}^T \\ H_{\sigma u_2}^T \\ H_{\sigma u_3}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{\sigma u_1}^T \\ \tilde{Q}_{\sigma u_2}^T \\ \tilde{Q}_{\sigma u_3}^T \end{bmatrix} - \begin{bmatrix} \check{Q}_{\sigma u_1}^T \\ \check{Q}_{\sigma u_2}^T \\ \check{Q}_{\sigma u_3}^T \end{bmatrix} \tag{VIII.278}$$

which has the solution

$$H_{\sigma u} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.279}$$

$$\check{Q}_{\sigma u} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.280}$$

where $H_{\sigma u} = 0$ by the fact that $\tilde{Q}_{\sigma u}$ was already in the proper normal form for $\check{Q}_{\sigma u}$.

Now let's recap the entire quadratic coordinate transformation. Putting all six pieces together, we have

$$h^{(2)}(\mu_1, z_1, z_2, y_1, y_2) = \begin{bmatrix} H_{\sigma u} & H_{\sigma m} & H_{\sigma c} \\ H_{yu} & H_{ym} & H_{yc} \end{bmatrix} \begin{bmatrix} \mu_1^2 \\ \mu_1 z \\ z^{(2)} \\ \mu_1 y_1 \\ z y_1 \\ \mu_1 y_2 \\ z y_2 \\ y^{(2)} \end{bmatrix} \tag{VIII.281}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{3}{4}z_1^2 + \frac{3}{4}z_2^2 + 24y_1^2 \end{bmatrix}.$$

The feedback law required to complete the transformation to quadratic normal form is found using the Non-Linear Feedback theorem in Chapter V. The feedback is given by the formula

$$\tilde{u} = v - \check{g}_\nu^{(1)}(\mu_1, z_1, z_2, y_1, y_2)\, v - \check{f}_\nu^{(2)}(\mu_1, z_1, z_2, y_1, y_2) \tag{VIII.282}$$

where $\check{f}_\nu^{(2)}(\mu_1, z_1, z_2, y_1, y_2)$ is the bottom row of $\check{f}^{(2)}(\mu_1, z_1, z_2, y_1, y_2)$, which can be calculated from $\check{Q}$, and $\check{g}_\nu^{(1)}(\mu_1, z_1, z_2, y_1, y_2)$ is the bottom row of $\check{g}^{(1)}(\mu_1, z_1, z_2, y_1, y_2)$,

which can be calculated from $\check{G}$. So, we calculate

$$\check{f}^{(2)}(\mu_1, z_1, z_2, y_1, y_2) = \begin{bmatrix} \check{Q}_{\sigma u} & \check{Q}_{\sigma m} & \check{Q}_{\sigma c} \\ \check{Q}_{yu} & \check{Q}_{ym} & \check{Q}_{yc} \end{bmatrix} \begin{bmatrix} \mu_1^2 \\ \mu_1 z \\ z^{(2)} \\ \mu_1 y_1 \\ z y_1 \\ \mu_1 y_2 \\ z y_2 \\ y^{(2)} \end{bmatrix} \tag{VIII.283}$$

$$= \begin{bmatrix} 0 \\ -\mu_1 z_1 - 24 z_1 y_1 \\ -\mu_1 z_2 - 24 z_2 y_1 \\ 0 \\ -52 y_1 y_2 - \frac{3}{2} y_2^2 \end{bmatrix}$$

and

$$\check{g}^{(1)}(\mu_1, z_1, z_2, y_1, y_2) = \begin{bmatrix} \check{G}_{\sigma u} & \check{G}_{\sigma c} \\ \check{G}_{yu} & \check{G}_{yc} \end{bmatrix} \begin{bmatrix} \mu_1 \\ z_1 \\ z_2 \\ y_1 \\ y_2 \end{bmatrix} \tag{VIII.284}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -y_2 \end{bmatrix}$$

which gives

$$\check{f}_{\nu}^{(2)}(\mu_1, z_1, z_2, y_1, y_2) = -52 y_1 y_2 - \frac{3}{2} y_2^2 \tag{VIII.285}$$

$$\check{g}_{\nu}^{(1)}(\mu_1, z_1, z_2, y_1, y_2) = -y_2 \tag{VIII.286}$$

which gives our feedback law as

$$\tilde{u} = v + y_2 v + 52 y_1 y_2 + \frac{3}{2} y_2^2 \tag{VIII.287}$$

Now with the quadratic coordinate change and non-linear state feedback law calculated, we can calculate the transformed cubic order terms. The cubic order terms are given by the equations

$$f_z^{(3)}(\mu_1, z_1, z_2, y_1, y_2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \left(\vec{f}^{(3)}(\chi) - \check{g}^{(1)}(\chi) \check{f}_{\nu}^{(2)}(\chi) \right) \tag{VIII.288}$$

$$g_z^{(2)}(\mu_1, z_1, z_2, y_1, y_2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \left(\vec{g}^{(2)}(\chi) - \check{g}^{(1)}(\chi) \check{g}_{\nu}^{(1)}(\chi) \right) \tag{VIII.289}$$

where

$$\begin{aligned} \check{g}^{(1)}(\chi) \check{f}_{\nu}^{(2)}(\chi) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -y_2 \end{bmatrix} \left(-52 y_1 y_2 - \frac{3}{2} y_2^2 \right) \qquad \text{(VIII.290)} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 52 y_1 y_2^2 + \frac{3}{2} y_2^3 \end{bmatrix} \end{aligned}$$

and where

$$\bar{f}^{(3)}(\chi) = \bar{f}^{(3)}(\chi) - \frac{\partial}{\partial \chi} h^{(2)}(\chi) \check{f}^{(2)}(\chi) \tag{VIII.291}$$

The term $\frac{\partial}{\partial \chi} h^{(2)}(\chi)$ is given by

$$\frac{\partial}{\partial \chi} h^{(2)}(\chi) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2}z_1 & \frac{3}{2}z_2 & 48y_1 & 0 \end{bmatrix} \tag{VIII.292}$$

so

$$\frac{\partial}{\partial \chi} h^{(2)}(\chi) \check{f}^{(2)}(\chi) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{3}{2}\mu_1 z_1^2 - \frac{3}{2}\mu_1 z_2^2 - 36 z_1^2 y_1 - 36 z_2^2 y_1 \end{bmatrix} \tag{VIII.293}$$

The term $\bar{f}^{(3)}(\chi)$ is defined by the relation

$$\bar{f}^{(3)}(\chi) + O^{(4+)} = \tilde{f}^{(3)}(\chi) + \left(\tilde{f}^{(2)}\left(\chi + h^{(2)}(\chi)\right) - \tilde{f}^{(2)}(\chi)\right) \tag{VIII.294}$$

where

$$\chi + h^{(2)}(\chi) = \begin{bmatrix} \mu_1 \\ z_1 \\ z_2 \\ y_1 \\ y_2 + \frac{3}{4}z_1^2 + \frac{3}{4}z_2^2 + 24y_1^2 \end{bmatrix} \tag{VIII.295}$$

and where

$$\tilde{f}^{(2)}(\mu_1, z_1, z_2, y_1, y_2) = \begin{bmatrix} 0 \\ -\mu_1 z_1 - 24 z_1 y_1 \\ -\mu_1 z_2 - 24 z_2 y_1 \\ -\frac{3}{4}z_1^2 - \frac{3}{4}z_2^2 - 24y_1^2 \\ -4y_1 y_2 - \frac{3}{2}y_2^2 \end{bmatrix} \tag{VIII.296}$$

So

$$\tilde{f}^{(2)}\left(\chi + h^{(2)}(\chi)\right) \tag{VIII.297}$$
$$= \begin{bmatrix} 0 \\ -\mu_1 z_1 - 24 z_1 y_1 \\ -\mu_1 z_2 - 24 z_2 y_1 \\ -\frac{3}{4}z_1^2 - \frac{3}{4}z_2^2 - 24y_1^2 \\ -4y_1\left(y_2 + \frac{3}{4}z_1^2 + \frac{3}{4}z_2^2 + 24y_1^2\right) - \frac{3}{2}\left(y_2 + \frac{3}{4}z_1^2 + \frac{3}{4}z_2^2 + 24y_1^2\right)^2 \end{bmatrix}$$

and

$$\tilde{f}^{(3)}(\mu_1, z_1, z_2, y_1, y_2) = \begin{bmatrix} 0 \\ -\frac{3}{4}z_1^3 - \frac{3}{4}z_1 z_2^2 - 48 z_1 y_1^2 \\ -\frac{3}{4}z_1^2 z_2 - \frac{3}{4}z_2^3 - 48 z_2 y_1^2 \\ -3z_1^2 y_1 - 3z_2^2 y_1 - 32y_1^3 \\ -2y_1 y_2^2 - y_2^3 \end{bmatrix} \tag{VIII.298}$$

So, we have

$$\bar{f}^{(3)}(\chi) + O^{(4+)} \tag{VIII.299}$$
$$= \begin{bmatrix} 0 \\ -\frac{3}{4}z_1^3 - \frac{3}{4}z_1 z_2^2 - 48 z_1 y_1^2 \\ -\frac{3}{4}z_1^2 z_2 - \frac{3}{4}z_2^3 - 48 z_2 y_1^2 \\ -3z_1^2 y_1 - 3z_2^2 y_1 - 32y_1^3 \\ -2y_1 y_2^2 - y_2^3 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -4y_1\left(y_2 + \frac{3}{4}z_1^2 + \frac{3}{4}z_2^2 + 24y_1^2\right) - \frac{3}{2}\left(y_2 + \frac{3}{4}z_1^2 + \frac{3}{4}z_2^2 + 24y_1^2\right)^2 + 4y_1 y_2 + \frac{3}{2}y_2^2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ -\frac{3}{4}z_1^3 - \frac{3}{4}z_1 z_2^2 - 48 z_1 y_1^2 \\ -\frac{3}{4}z_1^2 z_2 - \frac{3}{4}z_2^3 - 48 z_2 y_1^2 \\ -3z_1^2 y_1 - 3z_2^2 y_1 - 32y_1^3 \\ -2y_1 y_2^2 - y_2^3 - 4y_1\left(y_2 + \frac{3}{4}z_1^2 + \frac{3}{4}z_2^2 + 24y_1^2\right) - \frac{3}{2}\left(y_2 + \frac{3}{4}z_1^2 + \frac{3}{4}z_2^2 + 24y_1^2\right)^2 + 4y_1 y_2 + \frac{3}{2}y_2^2 \end{bmatrix}$$

which gives

$$f_z^{(3)}(\mu_1, z_1, z_2, y_1, y_2) = \begin{bmatrix} -\frac{3}{4}z_1^3 - \frac{3}{4}z_1 z_2^2 - 48 z_1 y_1^2 \\ -\frac{3}{4}z_1^2 z_2 - \frac{3}{4}z_2^3 - 48 z_2 y_1^2 \end{bmatrix} \qquad \text{(VIII.300)}$$

Now look at

$$g_z^{(2)}(\mu_1; z_1, z_2, y_1, y_2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \left(\vec{g}^{(2)}(\chi) - \check{g}^{(1)}(\chi)\,\check{g}_\nu^{(1)}(\chi)\right) \qquad \text{(VIII.301)}$$

where

$$\begin{aligned} \check{g}^{(1)}(\chi)\,\check{g}_\nu^{(1)}(\chi) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -y_2 \end{bmatrix} (-y_2) \qquad \text{(VIII.302)} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ y_2^2 \end{bmatrix} \end{aligned}$$

and

$$\vec{g}^{(2)}(\chi) - \check{g}^{(1)}(\chi)\,\check{g}_\nu^{(1)}(\chi) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{3}{4}z_1^2 - \frac{3}{4}z_2^2 - 24y_1^2 - \frac{3}{2}y_2^2 \end{bmatrix} \qquad \text{(VIII.303)}$$

So

$$g_z^{(2)}(\mu_1, z_1, z_2, y_1, y_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{VIII.304}$$

The quadratic normal form of our version of the four state Moore-Greitzer compressor model is

$$\dot{\mu}_1 = 0 \tag{VIII.305}$$

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &+ Q_{z_{P_1}} \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ \mu_1 z_2 \\ z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix} + Q_{z_{m_1}} \begin{bmatrix} \mu_1 y_1 \\ z_1 y_1 \\ z_2 y_1 \end{bmatrix} + Q_{z_c} \begin{bmatrix} y_1^2 \\ y_2^2 \end{bmatrix} \\ &+ f_z^{(3)}(\mu_1, z_1, z_2, y_1, y_2) + g_z^{(2)}(\mu_1, z_1, z_2, y_1, y_2)\, v + O^{(4+)} \end{aligned} \tag{VIII.306}$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v + Q_{y_c} \begin{bmatrix} y_1^2 \\ y_2^2 \end{bmatrix} + O^{(3+)} \tag{VIII.307}$$

where

$$Q_{z_{P_1}} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.308}$$

$$Q_{z_{m_1}} = \begin{bmatrix} 0 & -24 & 0 \\ 0 & 0 & -24 \end{bmatrix} \tag{VIII.309}$$

$$Q_{z_c} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{VIII.310}$$

$$Q_{y_c} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{VIII.311}$$

$$f_z^{(3)}(\mu_1, z_1, z_2, y_1, y_2) = \begin{bmatrix} -\frac{3}{4}z_1^3 - \frac{3}{4}z_1 z_2^2 - 48 z_1 y_1^2 \\ -\frac{3}{4}z_1^2 z_2 - \frac{3}{4}z_2^3 - 48 z_2 y_1^2 \end{bmatrix} \tag{VIII.312}$$

$$g_z^{(2)}(\mu_1, z_1, z_2, y_1, y_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{VIII.313}$$

5. Stabilize the Linearly Controllable States

Now that our system is in quadratic normal form, we want to use the techniques in Chapter VI to cause our linearly controllable states y to collapse onto the center manifold. We choose state feedback gains K_y such that the system is well damped, but still very responsive, and we desire a damping ratio of 0.707. So, if we pick $K_{y_1} = -2$ and $K_{y_2} = -2$, the closed loop eigenvalues of the system given by $A + BK_y^T$ are $\lambda = -1 \pm i$, which gives a damped frequency of $\omega_d = 1$ and a damping ratio of $\zeta = 0.707$, as desired.

6. Stabilize the Linearly Unstabilizable States

Now that we've stabilized the linearly controllable states y onto the center manifold, we need to use the techniques of Chapter VII to affect the bifurcation in a favorable manner. From the linear terms, the system exhibits a Hopf bifurcation, and we desire to affect the values of the coefficients a_0^*, α_1^* and ω_1^* in the polar coordinate equations

$$\dot{r} = \alpha_1^* \mu_1 r + a_0^* r^3 + H.O.T. \tag{VIII.314}$$

$$\dot{\theta} = \omega_0 + \omega_1^* \mu_1 + H.O.T. \tag{VIII.315}$$

Look at the terms α_1^* and ω_1^*. Lets assume that we want the origin to be stable when $\mu_1 < 0$, and that we don't want the frequency of the limit cycle to change with changes in μ_1, that is, we want

$$\alpha_1^* = 1 \tag{VIII.316}$$

$$\omega_1^* = 0 \tag{VIII.317}$$

So, we need to calculate the value of Π_L which forces the quadratic part of the dynamics of our system on the center manifold to take on these values. The equation we need to solve in general is

$$\begin{bmatrix} \left(q_{z_{m_{11}}} + 2q_{z_{c_{11}}}\Pi_{L_{11}}\right) & \left(q_{z_{m_{21}}} + 2q_{z_{c_{21}}}\Pi_{L_{11}}\right) \\ \left(q_{z_{m_{21}}} + 2q_{z_{c_{21}}}\Pi_{L_{11}}\right) & \left(q_{z_{m_{11}}} + 2q_{z_{c_{11}}}\Pi_{L_{11}}\right) \end{bmatrix} \begin{bmatrix} \Pi_{L_{12}} \\ \Pi_{L_{13}} \end{bmatrix} \tag{VIII.318}$$
$$= \begin{bmatrix} 2\alpha_1^* - \left(q_{z_{P_{12}}} + q_{z_{P_{23}}} + \left(q_{z_{m_{12}}} + q_{z_{m_{23}}}\right)\Pi_{L_{11}}\right) \\ 2\omega_1^* - \left(-q_{z_{P_{13}}} + q_{z_{P_{22}}} + \left(-q_{z_{m_{13}}} + q_{z_{m_{22}}}\right)\Pi_{L_{11}}\right) \end{bmatrix}$$

which for our specific system becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi_{L_{12}} \\ \Pi_{L_{13}} \end{bmatrix} = \begin{bmatrix} 2\alpha_1^* + (2 + 48\Pi_{L_{11}}) \\ 2\omega_1^* + 0 \end{bmatrix} \tag{VIII.319}$$

So, we find that we cannot control ω_1^* at all, and that our only influence on α_1^* is through $\Pi_{L_{11}}$. We have

$$\Pi_{L_{11}} = \frac{-\alpha_1^* - 1}{24} \tag{VIII.320}$$

$$\Pi_{L_{12}} = arbitrary \tag{VIII.321}$$

$$\Pi_{L_{13}} = arbitrary \tag{VIII.322}$$

and

$$\omega_1 = 0 \tag{VIII.323}$$

so we pick $\Pi_{L_{11}} = -\frac{1}{12}$ and $\Pi_{L_{12}} = \Pi_{L_{13}} = 0$. Now, to calculate the linear gains, we use the Hopf Bifurcation Linear Gains theorem in Chapter VII, which give the formulas

$$K_{\mu_1} = -K_{y_1}\Pi_{L_{11}} \tag{VIII.324}$$

$$K_{z_1} = G_A\Pi_{L_{12}} + G_B\Pi_{L_{13}} \tag{VIII.325}$$

$$K_{z_2} = G_A\Pi_{L_{13}} - G_B\Pi_{L_{12}} \tag{VIII.326}$$

which yield linear gain values of

$$K_{\mu_1} = -\frac{1}{6} \tag{VIII.327}$$

$$K_{z_1} = 0 \tag{VIII.328}$$

$$K_{z_2} = 0 \tag{VIII.329}$$

(Note that we did not have to calculate the values of G_A and G_B for this problem, although we could have done so. The formulas from the Hopf Bifurcation Linear Gains theorem are, for $p = 2$ and $\omega_0 = 1$,

$$\begin{aligned} G_A &= -1 - K_{y_1} \\ &= 1 \end{aligned} \tag{VIII.330}$$

$$\begin{aligned} G_B &= -K_{y_2} \\ &= 2 \end{aligned} \tag{VIII.331}$$

which we could have used if needed.)

Now we need to calculate the quadratic components of the center manifold, and the quadratic state feedback gains. First, we need to choose a desired value of a_0^*, and we know from equation VIII.314 that we require $a_0^* < 0$ for cubic stability, that is, to have the Hopf bifurcation be supercritical. Also, we know that the stabilized radius depends on both α_1^* and a_0^* through the relation

$$r^* = \sqrt{\frac{-\alpha_1^* \mu_1}{a_0^*}} \tag{VIII.332}$$

and since we have already chosen $\alpha_1^* = 1$, we choose $a_0^* = -9$ to keep the radius small. So, we can now calculate Π_Q. But first, let's calculate all of Π_L since we will need it later in the calculations. We have

$$\Pi_{L_1} = \begin{bmatrix} -\frac{1}{12} & 0 & 0 \end{bmatrix} \tag{VIII.333}$$

and

$$\Pi_{L_i} = \Pi_{L_1} \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^{i-1} \tag{VIII.334}$$

so for $i = 2$ we get

$$\Pi_{L_2} = \Pi_{L_1} \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} \tag{VIII.335}$$

$$= \begin{bmatrix} -\frac{1}{12} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

which gives the final answer as

$$\Pi_L = \begin{bmatrix} -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{VIII.336}$$

We will also need the value $\tilde{a}_0$, which is defined by the relation

$$\tilde{a}_0 = \frac{1}{8\omega_0} \left(q_{cm_{1_5}} \left(q_{cm_{1_4}} + q_{cm_{1_6}} \right) - q_{cm_{2_5}} \left(q_{cm_{2_4}} + q_{cm_{2_6}} \right) - 2 q_{cm_{1_4}} q_{cm_{2_4}} + 2 q_{cm_{1_6}} q_{cm_{2_6}} \right) \tag{VIII.337}$$

The coefficients $q_{cm_{i_j}}$ are given by the formulas

$$q_{cm_{1_4}} = q_{z_{P_{14}}} + q_{z_{m_{12}}} \Pi_{L_{12}} + \sum_{i=1}^{p} q_{z_{c_{1_i}}} \left(\Pi_{L_{i2}}\right)^2 \tag{VIII.338}$$

$$q_{cm_{1_5}} = q_{z_{P_{15}}} + q_{z_{m_{12}}} \Pi_{L_{13}} + q_{z_{m_{13}}} \Pi_{L_{12}} + 2 \sum_{i=1}^{p} q_{z_{c_{1_i}}} \Pi_{L_{i2}} \Pi_{L_{i3}} \tag{VIII.339}$$

$$q_{cm_{1_6}} = q_{z_{P_{16}}} + q_{z_{m_{13}}} \Pi_{L_{13}} + \sum_{i=1}^{p} q_{z_{c_{1_i}}} \left(\Pi_{L_{i3}}\right)^2 \tag{VIII.340}$$

$$q_{cm_{2_4}} = q_{z_{P_{24}}} + q_{z_{m_{22}}} \Pi_{L_{12}} + \sum_{i=1}^{p} q_{z_{c_{2_i}}} \left(\Pi_{L_{i2}}\right)^2 \tag{VIII.341}$$

$$q_{cm_{2_5}} = q_{z_{P_{25}}} + q_{z_{m_{22}}} \Pi_{L_{13}} + q_{z_{m_{23}}} \Pi_{L_{12}} + 2 \sum_{i=1}^{p} q_{z_{c_{2_i}}} \Pi_{L_{i2}} \Pi_{L_{i3}} \tag{VIII.342}$$

$$q_{cm_{2_6}} = q_{z_{P_{26}}} + q_{z_{m_{23}}} \Pi_{L_{13}} + \sum_{i=1}^{p} q_{z_{c_{2_i}}} \left(\Pi_{L_{i3}}\right)^2 \tag{VIII.343}$$

Now,

$$\tilde{a}_0 = 0 \tag{VIII.344}$$

because every $q_{cm_{ij}}$ is zero, since the only non-zero coefficient of Π_L is $\Pi_{L_{11}}$, and since $q_{zP_{12}}$ and $q_{zP_{23}}$ are the only non-zero coefficients of Q_{zP_1}. Thus, $\tilde{a}_0 = 0$.

We also need to calculate the matrix $\tilde{C}_z(\Pi_L)$ as a prelude to calculating Π_Q. From Chapter VII, the coefficient matrix $\tilde{C}_z(\Pi_L)$ for a two-dimensional co-dimension one bifurcation is defined by the relation

$$\tilde{C}_z(\Pi_L)\begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1^2 z_2 \\ \mu_1 z_1^2 \\ \mu_1 z_1 z_2 \\ \mu_1 z_2^2 \\ z_1^3 \\ z_1^2 z_2 \\ z_1 z_2^2 \\ z_2^3 \end{bmatrix} = \left(Q_{zP_2} N_3(\Omega_Q) + Q_{zm_2} N_5(\Pi_L, \Omega_Q)\right)\begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1^2 z_2 \\ \mu_1 z_1^2 \\ \mu_1 z_1 z_2 \\ \mu_1 z_2^2 \\ z_1^3 \\ z_1^2 z_2 \\ z_1 z_2^2 \\ z_2^3 \end{bmatrix} \qquad \text{(VIII.345)}$$
$$+ \; f_z^{(3)}\left(\mu_1, z_1, z_2, w_{cm}^{(1)}, y_{cm}^{(1)}\right) + g_z^{(2)}\left(\mu_1, z_1, z_2, w_{cm}^{(1)}, y_{cm}^{(1)}\right) v^{(1)}$$

For our system, we do not have any states w, so the term $w_{cm}^{(1)}$ and the matrices Q_{zP_2}, Q_{zm_2}, $N_3(\Omega_Q)$ and $N_5(\Pi_L, \Omega_Q)$ do not exist. (Note that even if we did have states w, the term $w_{cm}^{(1)}$ would always be zero, since $\Omega_L = 0$.) The term $y_{cm}^{(1)}$ is given by the relation

$$y_{cm}^{(1)} = \begin{bmatrix} y_{1_{cm}}^{(1)} \\ \vdots \\ y_{p_{cm}}^{(1)} \end{bmatrix} = \Pi_L \begin{bmatrix} \mu \\ z \end{bmatrix} \qquad \text{(VIII.346)}$$

For our system we have

$$\begin{bmatrix} y_{1_{cm}}^{(1)} \\ y_{2_{cm}}^{(1)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ z_1 \\ z_2 \end{bmatrix} \qquad \text{(VIII.347)}$$

$$= \begin{bmatrix} -\frac{1}{12}\mu_1 \\ 0 \end{bmatrix}$$

where we have plugged in equation VIII.336 for Π_L. Now we can calculate

$$f_z^{(3)}\left(\mu_1, z_1, z_2, w_{cm}^{(1)}, y_{cm}^{(1)}\right) = \begin{bmatrix} -\frac{3}{4}z_1^3 - \frac{3}{4}z_1 z_2^2 - 48 z_1 y_{1_{cm}}^{(1)2} \\ -\frac{3}{4}z_1^2 z_2 - \frac{3}{4}z_2^3 - 48 z_2 y_{1_{cm}}^{(1)2} \end{bmatrix} \tag{VIII.348}$$

$$= \begin{bmatrix} -\frac{1}{3}\mu_1^2 z_1 - \frac{3}{4}z_1^3 - \frac{3}{4}z_1 z_2^2 \\ -\frac{1}{3}\mu_1^2 z_2 - \frac{3}{4}z_1^2 z_2 - \frac{3}{4}z_2^3 \end{bmatrix}$$

where we have used equation VIII.312 from the quadratic normal form of our system. Now we can calculate

$$g_z^{(2)}\left(\mu_1, z_1, z_2, w_{cm}^{(1)}, y_{cm}^{(1)}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{VIII.349}$$

where we have used equation VIII.313 from the quadratic normal form of our system. We also need to calculate the term $v^{(1)}$ which includes the effect of the linear state feedback terms, which is

$$\begin{aligned} v^{(1)} &= K_\mu^T \mu + K_z^T z + K_w^T w_{cm}^{(1)} + K_y^T y_{cm}^{(1)} && \text{(VIII.350)} \\ &= K_{\mu_1}\mu_1 + K_{z_1} z_1 + K_{z_2} z_2 + K_{y_1} y_{1_{cm}}^{(1)} + K_{y_2} y_{2_{cm}}^{(1)} \\ &= \left(K_{\mu_1} - \frac{1}{12}K_{y_1}\right)\mu_1 + K_{z_1} z_1 + K_{z_2} z_2 \\ &= 0 \end{aligned}$$

where we have plugged in the linear state feedback gains

$$K_{\mu_1} = -\frac{1}{6} \tag{VIII.351}$$

$$K_{z_1} = 0 \tag{VIII.352}$$

$$K_{z_2} = 0 \tag{VIII.353}$$

$$K_{y_1} = -2 \tag{VIII.354}$$

$$K_{y_2} = -2 \tag{VIII.355}$$

from before. Plugging equations VIII.348, VIII.349 and VIII.350 into equation VIII.345 lets us calculate $\tilde{C}_z(\Pi_L)$, which is

$$\tilde{C}_z(\Pi_L) = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 & -\frac{3}{4} \end{bmatrix} \tag{VIII.356}$$

Finally, we can calculate the value of Π_Q which achieves the desired value of a_0^*. From Chapter VII the formula is

$$\Pi_{Q_1} = \left(\frac{\sigma}{V^T V}\right) V^T \tag{VIII.357}$$

where

$$\begin{aligned} \sigma &= 8\left(a_0^* - \tilde{a}_0\right) - \left(3\tilde{c}_{z1_7} + \tilde{c}_{z1_9} + \tilde{c}_{z2_8} + 3\tilde{c}_{z2_{10}}\right) \\ &\quad + 2\sum_{i=1}^{p}\sum_{j=0}^{i-1}\Gamma_{z_j}(\Pi_L)\begin{bmatrix} 0 \\ 0 \\ 0 \\ D_{zz}^{i-j-1}\begin{bmatrix} 3q_{zc_{1_i}}\Pi_{L_{i2}} + q_{zc_{2_i}}\Pi_{L_{i3}} \\ q_{zc_{1_i}}\Pi_{L_{i3}} + q_{zc_{2_i}}\Pi_{L_{i2}} \\ q_{zc_{1_i}}\Pi_{L_{i2}} + 3q_{zc_{2_i}}\Pi_{L_{i3}} \end{bmatrix} \end{bmatrix} \end{aligned} \tag{VIII.358}$$

and

$$V = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3q_{zm_{12}} + q_{zm_{23}} \\ q_{zm_{13}} + q_{zm_{22}} \\ q_{zm_{12}} + 3q_{zm_{23}} \end{bmatrix} + 2\sum_{i=1}^{p}\begin{bmatrix} 0 \\ 0 \\ 0 \\ D_{zz}^{i-1}\begin{bmatrix} 3q_{zc_{1_i}}\Pi_{L_{i2}} + q_{zc_{2_i}}\Pi_{L_{i3}} \\ q_{zc_{1_i}}\Pi_{L_{i3}} + q_{zc_{2_i}}\Pi_{L_{i2}} \\ q_{zc_{1_i}}\Pi_{L_{i2}} + 3q_{zc_{2_i}}\Pi_{L_{i3}} \end{bmatrix} \end{bmatrix} \tag{VIII.359}$$

where D_{zz} is obtained from Appendix A, and where

$$\Gamma_z(\Pi_L) = Q_{y_c} M_2(\Pi_L) - \Pi_L \begin{bmatrix} 0 \\ I \end{bmatrix} \left(Q_{zP_1} + Q_{zm_1} M_1(\Pi_L) + Q_{z_c} M_2(\Pi_L)\right) \tag{VIII.360}$$

Now, for our problem, since $Q_{z_c} = 0$ and since only $\Pi_{L_{11}}$ is non-zero in Π_L, we do not need to calculate either D_{zz} or $\Gamma_z(\Pi_L)$. However, for illustration purposes we have

$$D_{zz} = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix} \tag{VIII.361}$$

for a Hopf bifurcation from Appendix A, where we have plugged in $\omega_0 = 1$. To calculate $\Gamma_z(\Pi_L)$ we need to get the matrices $M_1(\Pi_L)$ and $M_2(\Pi_L)$ from Appendix C, which, for $p = 2$, are given by

$$\begin{aligned} M_1(\Pi_L) &= \begin{bmatrix} \Pi_{L_{11}} & \Pi_{L_{12}} & \Pi_{L_{13}} & 0 & 0 & 0 \\ 0 & \Pi_{L_{11}} & 0 & \Pi_{L_{12}} & \Pi_{L_{13}} & 0 \\ 0 & 0 & \Pi_{L_{11}} & 0 & \Pi_{L_{12}} & \Pi_{L_{13}} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{12} & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{VIII.362}$$

and

$$\begin{aligned} M_2(\Pi_L) &= \begin{bmatrix} (\Pi_{L_{11}})^2 & 2\Pi_{L_{11}}\Pi_{L_{12}} & 2\Pi_{L_{11}}\Pi_{L_{13}} & (\Pi_{L_{12}})^2 & 2\Pi_{L_{12}}\Pi_{L_{13}} & (\Pi_{L_{13}})^2 \\ (\Pi_{L_{21}})^2 & 2\Pi_{L_{21}}\Pi_{L_{22}} & 2\Pi_{L_{21}}\Pi_{L_{23}} & (\Pi_{L_{22}})^2 & 2\Pi_{L_{22}}\Pi_{L_{23}} & (\Pi_{L_{23}})^2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{144} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{VIII.363}$$

where we have plugged in equation VIII.336 for Π_L for our system. Now, we get the values for $Q_{z_{P_1}}$, $Q_{z_{m_1}}$, Q_{z_c} and Q_{y_c} from equations VIII.308 through VIII.311, which we can plug into equation VIII.360 to get

$$\Gamma_z(\Pi_L) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.364}$$

Now, plugging equation VIII.344 and the appropriate coefficients from the matrix $\tilde{C}_z(\Pi_L)$ in equation VIII.356, along with the desired value $a_0^* = -9$ into equation

$$D_{c2} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{A.38}$$

$$D_{c3} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.39}$$

for $p = 1$ through 3, where the subscript numeral indicates the dimension p in each case. For arbitrary p, the matrix D_c is given iteratively by the block formula

$$D_{c_{p+1}} = \begin{bmatrix} D_{c_p} & D_{B_y} & 0 \\ 0 & A & B \\ 0 & 0 & 0 \end{bmatrix} \tag{A.40}$$

where $D_{c_{p+1}} \in R^{\frac{(p+1)(p+2)}{2} \times \frac{(p+1)(p+2)}{2}}$ is the matrix for $y \in R^{p+1}$ which we are trying to calculate, $D_{c_p} \in R^{\frac{p(p+1)}{2} \times \frac{p(p+1)}{2}}$ is the matrix for $y \in R^p$ which we are assumed to know, and the coefficient matrices $D_{B_y} \in R^{\frac{p(p+1)}{2} \times p}$, $A \in R^{p \times p}$ and $B \in R^{p \times 1}$ were defined earlier.

Proof. We prove the cases for $p = 1$ through 3 by direct calculation, and prove the general case by examining the structure of $y^{(2)}$. For $y \in R^1$ we have $A = 0$. Since D_{c_1} is defined by the relation

$$D_{c_1} y_1^{(2)} \equiv \frac{\partial y_1^2}{\partial y_1} A y_1 = 0 \tag{A.41}$$

we have the trivial result that $D_{c_1} = 0$, thus proving case 1.

For $y \in R^2$ we have $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Since D_{c_2} is defined by the relation

$$D_{c2} \begin{bmatrix} y_1^2 \\ y_1 y_2 \\ y_2^2 \end{bmatrix} \equiv \frac{\partial \begin{bmatrix} y_1^2 \\ y_1 y_2 \\ y_2^2 \end{bmatrix}}{\partial \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2y_1 & 0 \\ y_2 & y_1 \\ 0 & 2y_2 \end{bmatrix} \begin{bmatrix} y_2 \\ 0 \end{bmatrix} \tag{A.42}$$

and

$$G_D = -\sum_{i=1}^{\frac{p}{2}} (-1)^{i-1} (2\omega_0)^{2i-1} \tilde{K}_{y_{2i}} \tag{VIII.372}$$
$$= -2K_{y_2}$$
$$= 4$$

where we have plugged in $K_{y_2} = -2$ from earlier, and used the definitions $\tilde{K}_{y_1} = 0$ and $\tilde{K}_{y_2} = K_{y_2}$. The coefficients Γ_{K_4}, Γ_{K_5} and Γ_{K_6} are found from the Hopf Gamma K Matrix lemma in Chapter VII using the formula

$$\begin{aligned}\left[\begin{array}{ccc} \Gamma_{K_4} & \Gamma_{K_5} & \Gamma_{K_6} \end{array}\right] &= -\sum_{j=0}^{p} \left[\begin{array}{ccc} \Gamma_{z_{j4}}(\Pi_L) & \Gamma_{z_{j5}}(\Pi_L) & \Gamma_{z_{j5}}(\Pi_L) \end{array}\right] D_{zz}^{p-j} \\ &+ \sum_{i=1}^{p}\sum_{j=0}^{i-1} \left[\begin{array}{ccc} \Gamma_{z_{j4}}(\Pi_L) & \Gamma_{z_{j5}}(\Pi_L) & \Gamma_{z_{j5}}(\Pi_L) \end{array}\right] D_{zz}^{i-j-1}\end{aligned} \tag{VIII.373}$$

where D_{zz} is found from equation VIII.361. Since $\Gamma_z(\Pi_L) = 0$ by equation VIII.364, we have $\Gamma_K = 0$ by the lemma. So, plugging in equations VIII.367, VIII.371 and VIII.372 and $\Gamma_{K_4} = \Gamma_{K_5} = \Gamma_{K_6} = 0$ into equations VIII.368, VIII.369 and VIII.370 gives

$$K_{z_1^2} = \frac{33}{48} \tag{VIII.374}$$

$$K_{z_1 z_2} = 0 \tag{VIII.375}$$

$$K_{z_2^2} = \frac{33}{48} \tag{VIII.376}$$

where we have used the fact that $K_{y_1} = -2$.

7. Undo the Transformations

Now that we have linear and quadratic state feedback gains which ensure a supercritical Hopf bifurcation with a small radius limit cycle in the transformed coordinate system, we need to reverse all the transformations to obtain the required control law in the original non-transformed system. This procedure was developed at the end of Chapter V and resulted in the formula

$$\begin{aligned} u &= \left(K_\chi^T - \left[\begin{array}{ccc} \alpha^T & 0 & a^T \end{array}\right]\right) T^{-1}\hat{\chi} \\ &- \left(G_\nu^T T^{-1}\hat{\chi}\right)\left(K_\chi^T T^{-1}\hat{\chi}\right) + \left(K_{\chi^{(2)}}^T - F_\nu^T - K_\chi^T H\right) \Upsilon \left(T^{-1}\right) \hat{\chi}^{(2)} \end{aligned} \tag{VIII.377}$$

where

$$\hat{\chi} = \begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{VIII.378}$$

and

$$\begin{aligned} K_{\chi}^T &= \begin{bmatrix} K_{\mu_1} & K_{z_1} & K_{z_2} & K_{y_1} & K_{y_2} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{6} & 0 & 0 & -2 & -2 \end{bmatrix} \end{aligned} \tag{VIII.379}$$

and

$$K_{\chi(2)}^T = \begin{bmatrix} K_{\sigma(2)}^T & K_{\sigma y(2)}^T & K_{y(2)}^T \end{bmatrix} \tag{VIII.380}$$

with

$$\begin{aligned} K_{\sigma(2)}^T &= \begin{bmatrix} K_{\mu_1^2} & K_{\mu_1 z_1} & K_{\mu_1 z_2} & K_{z_1^2} & K_{z_1 z_2} & K_{z_2^2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & \frac{33}{48} & 0 & \frac{33}{48} \end{bmatrix} \end{aligned} \tag{VIII.381}$$

$$\begin{aligned} K_{\sigma y(2)}^T &= \begin{bmatrix} K_{\mu_1 y_1} & K_{z_1 y_1} & K_{z_2 y_1} & K_{\mu_1 y_2} & K_{z_1 y_2} & K_{z_2 y_2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{VIII.382}$$

$$\begin{aligned} K_{y(2)}^T &= \begin{bmatrix} K_{y_1^2} & K_{y_1 y_2} & K_{y_2^2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{VIII.383}$$

From the linear coordinate transformation we have

$$\alpha^T = [0] \tag{VIII.384}$$

$$a^T = \begin{bmatrix} -4 & -1 \end{bmatrix} \tag{VIII.385}$$

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} \tag{VIII.386}$$

From the quadratic coordinate transformation we have

$$\begin{aligned} G_\nu^T &= \begin{bmatrix} \check{G}_{yu_p} & \check{G}_{yc_p} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{aligned} \tag{VIII.387}$$

and

$$F_\nu^T = \begin{bmatrix} \check{Q}_{yu_p} & \check{Q}_{ym_p} & \check{Q}_{yc_p} \end{bmatrix} \tag{VIII.388}$$

where

$$\check{Q}_{yu_p} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.389}$$

$$\check{Q}_{ym_p} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.390}$$

$$\check{Q}_{yc_p} = \begin{bmatrix} 0 & -52 & -\frac{3}{2} \end{bmatrix} \tag{VIII.391}$$

and

$$H = \begin{bmatrix} H_{\sigma u} & H_{\sigma m} & H_{\sigma c} \\ H_{yu} & H_{ym} & H_{yc} \end{bmatrix} \tag{VIII.392}$$

where

$$H_{\sigma u} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.393}$$

$$H_{\sigma m} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.394}$$

$$H_{\sigma c} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{VIII.395}$$

$$H_{yu} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{3}{4} \end{bmatrix} \tag{VIII.396}$$

$$H_{\sigma u} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{VIII.397}$$

$$H_{\sigma u} = \begin{bmatrix} 0 & 0 & 0 \\ 24 & 0 & 0 \end{bmatrix} \tag{VIII.398}$$

Finally, we have to calculate the coefficient matrix $\Upsilon\left(T^{-1}\right)$, which is defined by the relation

$$\tilde{\chi}^{(2)} = \Upsilon\left(T^{-1}\right)\hat{\chi}^{(2)} \tag{VIII.399}$$

where $\tilde{\chi} = T^{-1}\hat{\chi}$ with

$$\tilde{\chi} = \begin{bmatrix} \mu_1 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} \tag{VIII.400}$$

and

$$\hat{\chi} = \begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{VIII.401}$$

Different coefficient matrices $\Upsilon\left(T^{-1}\right)$ are possible depending on which component order is chosen for the quadratic state vector $\hat{\chi}^{(2)}$. If we choose

$$\hat{\chi}^{(2)} = \begin{bmatrix} \mu_1^2 \\ \mu_1 x_1 \\ \mu_1 x_2 \\ \mu_1 x_3 \\ \mu_1 x_4 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \\ x_1 x_3 \\ x_2 x_3 \\ x_3^2 \\ x_1 x_4 \\ x_2 x_4 \\ x_3 x_4 \\ x_4^2 \end{bmatrix} \tag{VIII.402}$$

then we have

$$\tilde{\chi}^{(2)} = \begin{bmatrix} \mu_1^2 \\ \mu_1\tilde{z}_1 \\ \mu_1\tilde{z}_2 \\ \tilde{z}_1^2 \\ \tilde{z}_1\tilde{z}_2 \\ \tilde{z}_2^2 \\ \mu_1\tilde{y}_1 \\ \tilde{z}_1\tilde{y}_1 \\ \tilde{z}_2\tilde{y}_1 \\ \mu_1\tilde{y}_2 \\ \tilde{z}_1\tilde{y}_2 \\ \tilde{z}_2\tilde{y}_2 \\ \tilde{y}_1^2 \\ \tilde{y}_1\tilde{y}_2 \\ \tilde{y}_2^2 \end{bmatrix} = \begin{bmatrix} \mu_1^2 \\ \mu_1 x_1 \\ \mu_1 x_2 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \\ \mu_1\left(\frac{1}{4}x_3\right) \\ x_1\left(\frac{1}{4}x_3\right) \\ x_2\left(\frac{1}{4}x_3\right) \\ \mu_1\left(-\frac{1}{2}x_4\right) \\ x_1\left(-\frac{1}{2}x_4\right) \\ x_2\left(-\frac{1}{2}x_4\right) \\ \left(\frac{1}{4}x_3\right)^2 \\ \left(\frac{1}{4}x_3\right)\left(-\frac{1}{2}x_4\right) \\ \left(-\frac{1}{2}x_4\right)^2 \end{bmatrix} = \begin{bmatrix} \mu_1^2 \\ \mu_1 x_1 \\ \mu_1 x_2 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \\ \frac{1}{4}\mu_1 x_3 \\ \frac{1}{4}x_1 x_3 \\ \frac{1}{4}x_2 x_3 \\ -\frac{1}{2}\mu_1 x_4 \\ -\frac{1}{2}x_1 x_4 \\ -\frac{1}{2}x_2 x_4 \\ \frac{1}{16}x_3^2 \\ -\frac{1}{8}x_3 x_4 \\ \frac{1}{4}x_4^2 \end{bmatrix} = \Upsilon\left(T^{-1}\right) \begin{bmatrix} \mu_1^2 \\ \mu_1 x_1 \\ \mu_1 x_2 \\ \mu_1 x_3 \\ \mu_1 x_4 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \\ x_1 x_3 \\ x_2 x_3 \\ x_3^2 \\ x_1 x_4 \\ x_2 x_4 \\ x_3 x_4 \\ x_4^2 \end{bmatrix} \tag{VIII.403}$$

so

$$
\Upsilon\left(T^{-1}\right) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}
\end{bmatrix} \tag{VIII.404}
$$

where we have used

$$
\begin{bmatrix} \mu_1 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = T^{-1} \begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \\ \frac{1}{4}x_3 \\ -\frac{1}{2}x_4 \end{bmatrix} \tag{VIII.405}
$$

Now we have to put all the parts together. We have

$$
\begin{aligned}
& K^T_{\chi^{(2)}} - F^T_\nu - K^T_\chi H \qquad \text{(VIII.406)} \\
&= \begin{bmatrix} 0 & 0 & 0 & \frac{105}{48} & 0 & \frac{105}{48} & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 52 & \frac{3}{2} \end{bmatrix}
\end{aligned}
$$

and

$$
\left(K^T_{\chi^{(2)}} - F^T_\nu - K^T_\chi H\right) \Upsilon\left(T^{-1}\right) \hat{\chi}^{(2)} \tag{VIII.407}
$$

$$= \frac{105}{48}x_1^2 + \frac{105}{48}x_2^2 + 3x_3^2 - \frac{13}{2}x_3x_4 + \frac{3}{8}x_4^2$$

Also

$$G_\nu^T T^{-1}\hat{\chi} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{VIII.408}$$

$$= \frac{1}{2}x_4$$

and

$$K_\chi^T T^{-1}\hat{\chi} = \begin{bmatrix} -\frac{1}{6} & 0 & 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \mu_1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{VIII.409}$$

$$= -\frac{1}{6}\mu_1 - \frac{1}{2}x_3 + x_4$$

so

$$\left(G_\nu^T T^{-1}\hat{\chi}\right)\left(K_\chi^T T^{-1}\hat{\chi}\right) = -\frac{1}{12}\mu_1 x_4 - \frac{1}{4}x_3x_4 + \frac{1}{2}x_4^2 \tag{VIII.410}$$

We also have

$$\begin{bmatrix} \alpha^T & 0 & a^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -4 & -1 \end{bmatrix} \tag{VIII.411}$$

so

$$K_\chi^T - \begin{bmatrix} \alpha^T & 0 & a^T \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & 0 & 0 & 2 & -1 \end{bmatrix} \tag{VIII.412}$$

and

$$\left(K_\chi^T - \begin{bmatrix} \alpha^T & 0 & a^T \end{bmatrix}\right) T^{-1}\hat{\chi} = -\frac{1}{6}\mu_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \tag{VIII.413}$$

Putting all the pieces together gives

$$\begin{aligned} u &= -\frac{1}{6}\mu_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ &+ \frac{1}{12}\mu_1 x_4 + \frac{105}{48}x_1^2 + \frac{105}{48}x_2^2 + 3x_3^2 - \frac{25}{4}x_3x_4 - \frac{1}{8}x_4^2 \end{aligned} \tag{VIII.414}$$

where

$$\mu_1 = \mu - \Psi_C'(\Phi^*) \quad \text{(VIII.415)}$$

$$x_1 = A_1 \quad \text{(VIII.416)}$$

$$x_2 = B_1 \quad \text{(VIII.417)}$$

$$x_3 = \Phi - \Phi^* \quad \text{(VIII.418)}$$

$$x_4 = \Psi - \Psi_C(\Phi^*) \quad \text{(VIII.419)}$$

$$u = \gamma - \frac{\Phi^*}{\sqrt{\Psi_C(\Phi^*)}} \quad \text{(VIII.420)}$$

with

$$\Phi^* = 1 \quad \text{(VIII.421)}$$

$$\Psi_C(\Phi^*) = 1 \quad \text{(VIII.422)}$$

$$\Psi_C'(\Phi^*) = 0 \quad \text{(VIII.423)}$$

Normally, equation VIII.414 suffices for a control law. However, if the control law is needed in the original variables, the coordinate translations can be plugged into equation VIII.414 giving

$$\begin{aligned} \gamma &= -\frac{27}{8} - \frac{1}{4}\mu + \frac{3}{4}\Phi + 7\Psi \\ &+ \frac{1}{12}\mu\Psi + \frac{105}{48}\left(A_1^2 + B_1^2\right) + 3\Phi^2 - \frac{25}{4}\Phi\Psi - \frac{1}{8}\Psi^2 \end{aligned} \quad \text{(VIII.424)}$$

as an alternate control law.

APPENDIX A. STRUCTURAL MATRICES USED IN QUADRATIC COORDINATE TRANSFORMATIONS

There are several matrices which occur in the course of performing quadratic coordinate transformations which do not depend strongly on the details of the problem being considered. Instead, they occur as a consequence of the structure of the solution method being used, and consequently will be referred to as "structural" matrices here and in the text. These matrices include the constant coefficient matrices D_{B_y}, D_{B_σ}, D_c, D_m, D_σ, D_ξ, D_ρ $D_{\mu z}$, D_η, D_{zz}, and if appropriate D_w, and D_{u_m}.

Where do these matrices come from? They are a consequence of the fact that the linear terms in our system have a fundamental structure, which we imposed on our system when we performed our linear coordinate transformation and linear state feedback. This structure is seen in the block diagonal nature of the linear part of our system, i.e.

$$\begin{bmatrix} \dot{\sigma} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} F_\sigma & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \sigma \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u + O^{(2+)} \tag{A.1}$$

where $y \in R^p$ is the vector of linearly uncontrollable states, $\sigma \in R^s$ is the vector of linearly controllable states, and the matrices A and B have the following structure:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & \cdots & & 0 \end{bmatrix} \tag{A.2}$$

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{A.3}$$

Additionally the vector σ and the matrix F_σ have the following block internal structures:

$$\sigma = \begin{bmatrix} \mu \\ z \\ w \end{bmatrix} \tag{A.4}$$

$$F_\sigma = \begin{bmatrix} 0 & 0 & 0 \\ F_\mu & F_z & 0 \\ 0 & 0 & F_w \end{bmatrix} \tag{A.5}$$

where $\mu \in R^r$ is the vector of parameters, $z \in R^q$ is the vector of linearly uncontrollable states having eigenvalues with zero real parts, $w \in R^m$ is the vector of linearly uncontrollable states having eigenvalues with non-zero real parts, and F_μ, F_z, and F_w are coefficient matrices corresponding to the appropriate vectors.

Now, if we look at our quadratic coordinate transformation, we have

$$\tilde{\chi} = \chi + h^{(2)}(\chi) \tag{A.6}$$

where we have defined

$$\chi = \begin{bmatrix} \mu \\ z \\ w \\ y \end{bmatrix} \tag{A.7}$$

and where we can define $h^{(2)}(\chi)$ in vector/matrix form as

$$h^{(2)}(\chi) = H\left[\chi^{(2)}\right] \tag{A.8}$$

with H a matrix of coefficients, and $\chi^{(2)}$ the vector of all possible quadratic combinations of the components of χ. Then, when we apply our quadratic coordinate transformation to our system, we end up trying to solve an homological equation of the form

$$f^{(2)}(\chi) + \begin{bmatrix} F_\sigma & 0 \\ 0 & A \end{bmatrix} h^{(2)}(\chi) - \frac{\partial h^{(2)}(\chi)}{\partial \chi} \begin{bmatrix} F_\sigma & 0 \\ 0 & A \end{bmatrix} \chi = 0 \tag{A.9}$$

with a constraint equation of the form

$$g^{(1)}(\chi) - \frac{\partial h^{(2)}(\chi)}{\partial \chi}\begin{bmatrix} 0 \\ B \end{bmatrix} = 0 \tag{A.10}$$

In this appendix we are interested in the terms in each equation which contain the derivative, which can be written as

$$\frac{\partial h^{(2)}(\chi)}{\partial \chi}\begin{bmatrix} F_\sigma & 0 \\ 0 & A \end{bmatrix}\chi = H\frac{\partial \chi^{(2)}}{\partial \chi}\begin{bmatrix} F_\sigma & 0 \\ 0 & A \end{bmatrix}\chi \tag{A.11}$$

and

$$\frac{\partial h^{(2)}(\chi)}{\partial \chi}\begin{bmatrix} 0 \\ B \end{bmatrix} = H\frac{\partial \chi^{(2)}}{\partial \chi}\begin{bmatrix} 0 \\ B \end{bmatrix} \tag{A.12}$$

Now, since our state vector χ has internal structure

$$\chi = \begin{bmatrix} \sigma \\ y \end{bmatrix} \tag{A.13}$$

where the vector σ is the linearly uncontrollable states grouped together, and the vector y is the linearly controllable states grouped together, we can impose a structure on the quadratic state vector $\chi^{(2)}$ as well, i.e.

$$\chi^{(2)} = \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix} \tag{A.14}$$

where $\sigma^{(2)}$ indicates the grouping of all quadratic state components having only terms from the linearly uncontrollable states, σ; where $\sigma y^{(2)}$ indicates the grouping of all quadratic state components having mixed terms from both the linearly uncontrollable and linearly controllable states; and where $y^{(2)}$ indicates the grouping of all quadratic state components having only terms from the linearly controllable states, y. Grouping

terms in this fashion and taking the derivative gives us

$$\frac{\partial \chi^{(2)}}{\partial \chi} = \frac{\partial \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix}}{\partial \begin{bmatrix} \sigma \\ y \end{bmatrix}} = \begin{bmatrix} \frac{\partial \sigma^{(2)}}{\partial \sigma} & \frac{\partial \sigma^{(2)}}{\partial y} \\ \frac{\partial \sigma y^{(2)}}{\partial \sigma} & \frac{\partial \sigma y^{(2)}}{\partial y} \\ \frac{\partial y^{(2)}}{\partial \sigma} & \frac{\partial y^{(2)}}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial \sigma^{(2)}}{\partial \sigma} & 0 \\ \frac{\partial \sigma y^{(2)}}{\partial \sigma} & \frac{\partial \sigma y^{(2)}}{\partial y} \\ 0 & \frac{\partial y^{(2)}}{\partial y} \end{bmatrix} \tag{A.15}$$

where the two opposite diagonal terms are identically zero since there are no components of y in $\sigma^{(2)}$, and no components of σ in $y^{(2)}$.

Now we can multiply out the terms containing the derivative and get

$$\begin{aligned} \frac{\partial \chi^{(2)}}{\partial \chi} \begin{bmatrix} F_\sigma & 0 \\ 0 & A \end{bmatrix} \chi &= \begin{bmatrix} \frac{\partial \sigma^{(2)}}{\partial \sigma} & 0 \\ \frac{\partial \sigma y^{(2)}}{\partial \sigma} & \frac{\partial \sigma y^{(2)}}{\partial y} \\ 0 & \frac{\partial y^{(2)}}{\partial y} \end{bmatrix} \begin{bmatrix} F_\sigma & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \sigma \\ y \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial \sigma^{(2)}}{\partial \sigma} F_\sigma \sigma \\ \frac{\partial \sigma y^{(2)}}{\partial \sigma} F_\sigma \sigma + \frac{\partial \sigma y^{(2)}}{\partial y} A y \\ \frac{\partial y^{(2)}}{\partial y} A y \end{bmatrix} \\ &= \begin{bmatrix} D_\sigma & 0 & 0 \\ 0 & D_m & 0 \\ 0 & 0 & D_c \end{bmatrix} \begin{bmatrix} \sigma^{(2)} \\ \sigma y^{(2)} \\ y^{(2)} \end{bmatrix} \end{aligned} \tag{A.16}$$

and

$$\begin{aligned} \frac{\partial \chi^{(2)}}{\partial \chi} \begin{bmatrix} 0 \\ B \end{bmatrix} &= \begin{bmatrix} \frac{\partial \sigma^{(2)}}{\partial \sigma} & 0 \\ \frac{\partial \sigma y^{(2)}}{\partial \sigma} & \frac{\partial \sigma y^{(2)}}{\partial y} \\ 0 & \frac{\partial y^{(2)}}{\partial y} \end{bmatrix} \begin{bmatrix} 0 \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\partial \sigma y^{(2)}}{\partial y} B \\ \frac{\partial y^{(2)}}{\partial y} B \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ D_{B_\sigma} & 0 \\ 0 & D_{B_y} \end{bmatrix} \begin{bmatrix} \sigma \\ y \end{bmatrix} \end{aligned} \tag{A.17}$$

where we have defined the coefficient matrices D_σ, D_m, D_c, D_{B_σ} and D_{B_y} by the relations

$$D_\sigma \sigma^{(2)} \equiv \frac{\partial \sigma^{(2)}}{\partial \sigma} F_\sigma \sigma \tag{A.18}$$

$$D_m \sigma y^{(2)} \equiv \frac{\partial \sigma y^{(2)}}{\partial \sigma} F_\sigma \sigma + \frac{\partial \sigma y^{(2)}}{\partial y} A y \tag{A.19}$$

$$D_c y^{(2)} \equiv \frac{\partial y^{(2)}}{\partial y} A y \tag{A.20}$$

$$D_{B_\sigma} \sigma \equiv \frac{\partial \sigma y^{(2)}}{\partial y} B \tag{A.21}$$

$$D_{B_y} y \equiv \frac{\partial y^{(2)}}{\partial y} B \tag{A.22}$$

Notice that these matrices don't have a heavy dependence on the specifics of a given problem — they are only dependent on the dimension and linear terms of our system, which is already in simplified form.

Now we will begin our calculations. We start with D_{B_y} and work our way upwards.

1. CALCULATION OF D_{B_Y}

Theorem 1.1 (Structure of D_{B_y}) *For any linearly controllable state vector* $y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} \in R^p$, *with input coefficient matrix* $B \in R^{p \times 1}$ *of the form*

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{A.23}$$

and with linearly controllable quadratic state vector $y^{(2)} \in R^{\frac{p(p+1)}{2}}$ *arranged as follows*

$$y^{(2)} = \begin{bmatrix} y_1^2 \\ y_1 y_2 \\ y_2^2 \\ y_1 y_3 \\ y_2 y_3 \\ y_3^2 \\ \vdots \\ y_p^2 \end{bmatrix} \tag{A.24}$$

then the matrix $D_{B_y} \in R^{\frac{p(p+1)}{2} \times p}$*, defined by the relation*

$$D_{B_y} y \equiv \frac{\partial y^{(2)}}{\partial y} B \tag{A.25}$$

is given in block form by

$$D_{B_y} = \begin{bmatrix} 0 \\ I_2 \end{bmatrix} \tag{A.26}$$

where the matrix $I_2 \in R^{p \times p}$ *is defined as*

$$I_2 = \begin{bmatrix} I & 0 \\ 0 & 2 \end{bmatrix} \tag{A.27}$$

where the lower right hand element is a scalar 2.

Proof. When the term $\frac{\partial y^{(2)}}{\partial y} B$ is calculated, only the last p terms are non-zero, since they are the only terms containing y_p. We get

$$\frac{\partial y^{(2)}}{\partial y} B = \begin{bmatrix} \frac{\partial y^{(2)}}{\partial y_1} & \cdots & \frac{\partial y^{(2)}}{\partial y_p} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \frac{\partial y^{(2)}}{\partial y_p} \tag{A.28}$$

$$= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ y_1 \\ \vdots \\ y_{p-1} \\ 2y_p \end{bmatrix} = \begin{bmatrix} 0 \\ I_2 \end{bmatrix} y$$

which proves the theorem. ◁

2. CALCULATION OF D_{B_σ}

Theorem 2.1 (Structure of D_{B_σ}) *For any linearly controllable state vector* $y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} \in R^p$ *and linearly uncontrollable state vector* $\sigma = \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_2 \end{bmatrix} \in R^s$*, with input coefficient matrix* $B \in R^{p\times 1}$ *of the form*

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{A.29}$$

and with mixed quadratic state vector $\sigma y^{(2)} \in R^{ps}$ *arranged in block form as*

$$\sigma y^{(2)} = \begin{bmatrix} \sigma y_1 \\ \vdots \\ \sigma y_p \end{bmatrix} \tag{A.30}$$

then the matrix $D_{B_\sigma} \in R^{ps\times s}$*, defined by the relation*

$$D_{B_\sigma}\sigma \equiv \frac{\partial \sigma y^{(2)}}{\partial y} B \tag{A.31}$$

is given in block form by

$$D_{B_\sigma} = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{A.32}$$

where $I \in R^{s\times s}$ *is the identity matrix.*

Proof. When the term $\frac{\partial \sigma y^{(2)}}{\partial y} B$ is calculated, only the last s terms are non-zero, since they are the only terms containing y_p. We get

$$\frac{\partial \sigma y^{(2)}}{\partial y} B = \begin{bmatrix} \frac{\partial \sigma y^{(2)}}{\partial y_1} & \cdots & \frac{\partial \sigma y^{(2)}}{\partial y_p} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \frac{\partial \sigma y^{(2)}}{\partial y_p} \tag{A.33}$$

$$
= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sigma_1 \\ \vdots \\ \sigma_p \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} \sigma
$$

which proves the theorem. ◁

3. CALCULATION OF D_C

Theorem 3.1 (Structure of D_c) *For any linear state vector* $y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} \in R^p$, *with coefficient matrix* $A \in R^{p \times p}$ *of the form*

$$
A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & \cdots & & 0 \end{bmatrix} \tag{A.34}
$$

and with controllable states quadratic state vector $y^{(2)} \in R^{\frac{p(p+1)}{2}}$ *arranged as follows*

$$
y^{(2)} = \begin{bmatrix} y_1^2 \\ y_1 y_2 \\ y_2^2 \\ y_1 y_3 \\ y_2 y_3 \\ y_3^2 \\ \vdots \\ y_p^2 \end{bmatrix} \tag{A.35}
$$

then the matrix D_c, *defined by the relation*

$$
D_c y^{(2)} \equiv \frac{\partial y^{(2)}}{\partial y} A y \tag{A.36}
$$

is given by

$$
D_{c_1} = [0] \tag{A.37}
$$

$$D_{c_2} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{A.38}$$

$$D_{c_3} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.39}$$

for $p = 1$ through 3, where the subscript numeral indicates the dimension p in each case. For arbitrary p, the matrix D_c is given iteratively by the block formula

$$D_{c_{p+1}} = \begin{bmatrix} D_{c_p} & D_{B_y} & 0 \\ 0 & A & B \\ 0 & 0 & 0 \end{bmatrix} \tag{A.40}$$

where $D_{c_{p+1}} \in R^{\frac{(p+1)(p+2)}{2} \times \frac{(p+1)(p+2)}{2}}$ is the matrix for $y \in R^{p+1}$ which we are trying to calculate, $D_{c_p} \in R^{\frac{p(p+1)}{2} \times \frac{p(p+1)}{2}}$ is the matrix for $y \in R^p$ which we are assumed to know, and the coefficient matrices $D_{B_y} \in R^{\frac{p(p+1)}{2} \times p}$, $A \in R^{p \times p}$ and $B \in R^{p \times 1}$ were defined earlier.

Proof. We prove the cases for $p = 1$ through 3 by direct calculation, and prove the general case by examining the structure of $y^{(2)}$. For $y \in R^1$ we have $A = 0$. Since D_{c_1} is defined by the relation

$$D_{c_1} y_1^{(2)} \equiv \frac{\partial y_1^2}{\partial y_1} A y_1 = 0 \tag{A.41}$$

we have the trivial result that $D_{c_1} = 0$, thus proving case 1.

For $y \in R^2$ we have $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Since D_{c_2} is defined by the relation

$$D_{c_2} \begin{bmatrix} y_1^2 \\ y_1 y_2 \\ y_2^2 \end{bmatrix} \equiv \frac{\partial \begin{bmatrix} y_1^2 \\ y_1 y_2 \\ y_2^2 \end{bmatrix}}{\partial \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2y_1 & 0 \\ y_2 & y_1 \\ 0 & 2y_2 \end{bmatrix} \begin{bmatrix} y_2 \\ 0 \end{bmatrix} \tag{A.42}$$

$$= \begin{bmatrix} 2y_1y_2 \\ y_2^2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1^2 \\ y_1y_2 \\ y_2^2 \end{bmatrix}$$

we have $D_{c_2} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, thus proving case 2.

For $y \in R^3$ we have $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Since D_{c_3} is defined by the relation

$$D_{c_3} \begin{bmatrix} y_1^2 \\ y_1y_2 \\ y_2^2 \\ y_1y_3 \\ y_2y_3 \\ y_3^2 \end{bmatrix} \equiv \frac{\partial \begin{bmatrix} y_1^2 \\ y_1y_2 \\ y_2^2 \\ y_1y_3 \\ y_2y_3 \\ y_3^2 \end{bmatrix}}{\partial \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} A \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2y_1 & 0 & 0 \\ y_2 & y_1 & 0 \\ 0 & 2y_2 & 0 \\ y_3 & 0 & y_1 \\ 0 & y_3 & y_2 \\ 0 & 0 & 2y_3 \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ 0 \end{bmatrix} \tag{A.43}$$

$$= \begin{bmatrix} 2y_1y_2 \\ y_2^2 + y_1y_3 \\ 2y_2y_3 \\ y_2y_3 \\ y_3^2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1^2 \\ y_1y_2 \\ y_2^2 \\ y_1y_3 \\ y_2y_3 \\ y_3^2 \end{bmatrix}$$

we have $D_{c_3} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, thus proving case 3.

For the general case, we examine the structure of $y^{(2)}$. When we increase the dimension of the linearly controllable state vector y from p to $p+1$, the linear state vector becomes

$$y_{new} = \begin{bmatrix} y_{old} \\ y_{p+1} \end{bmatrix} \tag{A.44}$$

Likewise, the quadratic state vector becomes

$$y^{(2)}_{new} = \begin{bmatrix} y^{(2)}_{old} \\ y_{old}y_{p+1} \\ y^2_{p+1} \end{bmatrix} \tag{A.45}$$

Here we will drop the subscripts $_{old}$ and $_{new}$ with the assumption that all references to the state vectors y and $y^{(2)}$ refer to the dimension p, rather than $p+1$. We have $D_c y^{(2)} \equiv \frac{\partial y^{(2)}}{\partial y} Ay$, which yields

$$\begin{aligned} D_{c_{p+1}} \begin{bmatrix} y^{(2)} \\ yy_{p+1} \\ y^2_{p+1} \end{bmatrix} &= \frac{\partial \begin{bmatrix} y^{(2)} \\ yy_{p+1} \\ y^2_{p+1} \end{bmatrix}}{\partial \begin{bmatrix} y \\ y_{p+1} \end{bmatrix}} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ y_{p+1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial y^{(2)}}{\partial y} & 0 \\ Iy_{p+1} & y \\ 0 & 2 \end{bmatrix} \begin{bmatrix} Ay + By_{p+1} \\ 0 \end{bmatrix} \end{aligned} \tag{A.46}$$

$$= \begin{bmatrix} \frac{\partial y^{(2)}}{\partial y} A y + \frac{\partial y^{(2)}}{\partial y} B y_{p+1} \\ A y y_{p+1} + B y_{p+1}^2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} D_{c_p} y^{(2)} + D_{B_y} y y_{p+1} \\ A y y_{p+1} + B y_{p+1}^2 \\ 0 \end{bmatrix}$$

where we have used the fact that when p increases to $p+1$ in the first equation, the new A matrix changes to

$$A_{p+1} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \tag{A.47}$$

and where we have used equations A.20 and A.22 for the definitions of the matrices D_{c_p} and D_{B_y}. Finally, pulling out the new quadratic state vector yields

$$D_{c_{p+1}} \begin{bmatrix} y^{(2)} \\ y y_{p+1} \\ y_{p+1}^2 \end{bmatrix} = \begin{bmatrix} D_{c_p} & D_{B_y} & 0 \\ 0 & A & B \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y^{(2)} \\ y y_{p+1} \\ y_{p+1}^2 \end{bmatrix}. \tag{A.48}$$

which proves the general case for the theorem. $\triangleleft$

4. CALCULATION OF D_M

Theorem 4.1 (Structure of D_m) *For any linear state vector* $\chi = \begin{bmatrix} \sigma \\ y \end{bmatrix}$, *with* $\sigma \in R^s$ *and* $y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} \in R^p$, *and with controllable state coefficient matrix* $A \in R^{p \times p}$ *of the form*

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & \cdots & & 0 \end{bmatrix} \tag{A.49}$$

and with uncontrollable state coefficient matrix $F_\sigma \in R^{s\times s}$ *and with mixed quadratic state vector* $\sigma y^{(2)} \in R^{ps}$ *arranged as follows*

$$\sigma y^{(2)} = \begin{bmatrix} \sigma y_1 \\ \sigma y_2 \\ \vdots \\ \sigma y_p \end{bmatrix} = \begin{bmatrix} \sigma_1 y_1 \\ \sigma_2 y_1 \\ \vdots \\ \sigma_s y_1 \\ \vdots \\ \sigma_1 y_p \\ \sigma_2 y_p \\ \vdots \\ \sigma_s y_p \end{bmatrix} \tag{A.50}$$

then the matrix $D_m \in R^{ps\times ps}$, *defined by the relation*

$$D_m \sigma y^{(2)} \equiv \frac{\partial \sigma y^{(2)}}{\partial \sigma} F_\sigma \sigma + \frac{\partial \sigma y^{(2)}}{\partial y} A y \tag{A.51}$$

is given in block form by

$$D_m = \begin{bmatrix} F_\sigma & I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I \\ 0 & \cdots & \cdots & 0 & F_\sigma \end{bmatrix} \tag{A.52}$$

Proof. We prove the general case by exploiting the structure of $\sigma y^{(2)}$. We calculate

$$\frac{\partial \sigma y^{(2)}}{\partial \sigma} F_\sigma \sigma = \frac{\partial \begin{bmatrix} \sigma y_1 \\ \vdots \\ \sigma y_p \end{bmatrix}}{\partial \sigma} F_\sigma \sigma = \begin{bmatrix} y_1 I \\ \vdots \\ y_p I \end{bmatrix} F_\sigma \sigma = \begin{bmatrix} F_\sigma \sigma y_1 \\ \vdots \\ F_\sigma \sigma y_p \end{bmatrix} \tag{A.53}$$

and

$$\frac{\partial \sigma y^{(2)}}{\partial y} A y = \frac{\partial \begin{bmatrix} \sigma y_1 \\ \vdots \\ \sigma y_p \end{bmatrix}}{\partial y} A y = \begin{bmatrix} \sigma & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma \end{bmatrix} \begin{bmatrix} y_2 \\ \vdots \\ y_p \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma y_2 \\ \vdots \\ \sigma y_p \\ 0 \end{bmatrix} \tag{A.54}$$

which we put together to get

$$D_\sigma \sigma y^{(2)} \equiv \frac{\partial \sigma y^{(2)}}{\partial \sigma} F_\sigma \sigma + \frac{\partial \sigma y^{(2)}}{\partial y} A y = \begin{bmatrix} F_\sigma \sigma y_1 \\ \vdots \\ F_\sigma \sigma y_{p-1} \\ F_\sigma \sigma y_p \end{bmatrix} + \begin{bmatrix} \sigma y_2 \\ \vdots \\ \sigma y_p \\ 0 \end{bmatrix} \tag{A.55}$$

$$= \begin{bmatrix} F_\sigma & I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I \\ 0 & \cdots & \cdots & 0 & F_\sigma \end{bmatrix} \begin{bmatrix} \sigma y_1 \\ \sigma y_2 \\ \sigma y_3 \\ \vdots \\ \sigma y_p \end{bmatrix}$$

which proves the general case. $\triangleleft$

5. CALCULATION OF D_σ

We now look at the calculation of D_σ. The matrix D_σ is used in the calculation of Poincare normal forms for general uncontrolled dynamic systems. For this dissertation its primary use is as a way to account for the effect of any linearly uncontrollable but stable states w and to allow us to get to the matrix D_ξ, which as we will see, is of use calculating the Poincare normal forms for the bifurcations of interest, and in the control of the center manifold of dynamic systems. The calculation of D_σ is complicated by the fact that the linear vector of uncontrollable states, σ, is made up of three sub-vectors of interest: μ, the vector of parameters; z, the vector of linearly uncontrollable states having zero real-part eigenvalues; and w, the vector of linearly uncontrollable states having non-zero real-part eigenvalues. We will group the center manifold states as $\xi = \begin{bmatrix} \mu \\ z \end{bmatrix}$, which will allow us to consider D_σ to be made up of three separate sub-matrices: D_ξ, D_{u_m} and D_w. Then in a subsequent sections we will consider the structure of these sub-matrices.

Theorem 5.1 (General Structure of D_σ) *For any vector of linearly uncontrollable states σ given in block form by $\sigma = \begin{bmatrix} \xi \\ w \end{bmatrix}$, and with linearly uncontrollable state coefficient matrix F_σ having the block form*

$$F_\sigma = \begin{bmatrix} F_\xi & 0 \\ 0 & F_w \end{bmatrix} \tag{A.56}$$

the structural matrix D_σ, defined by the relation

$$D_\sigma \sigma^{(2)} \equiv \frac{\partial \sigma^{(2)}}{\partial \sigma} F_\sigma \sigma \tag{A.57}$$

has the block form

$$D_\sigma = \begin{bmatrix} D_\xi & 0 & 0 \\ 0 & D_{u_m} & 0 \\ 0 & 0 & D_w \end{bmatrix} \tag{A.58}$$

where the sub-matrices D_ξ, D_{u_m} and D_w are defined by the relations

$$D_\xi \xi^{(2)} \equiv \frac{\partial \xi^{(2)}}{\partial \xi} F_\xi \xi \tag{A.59}$$

$$D_{u_m} \xi w^{(2)} \equiv \frac{\partial \xi w^{(2)}}{\partial \xi} F_\xi \xi + \frac{\partial \xi w^{(2)}}{\partial w} F_w w \tag{A.60}$$

$$D_w w^{(2)} \equiv \frac{\partial w^{(2)}}{\partial w} F_w w \tag{A.61}$$

Proof. Since $\sigma = \begin{bmatrix} \xi \\ w \end{bmatrix}$, the quadratic state vector has the form

$$\sigma^{(2)} = \begin{bmatrix} \xi^{(2)} \\ \xi w^{(2)} \\ w^{(2)} \end{bmatrix} \tag{A.62}$$

So

$$\begin{aligned} D_\sigma \sigma^{(2)} &\equiv \frac{\partial \sigma^{(2)}}{\partial \sigma} F_\sigma \sigma \\ &= \begin{bmatrix} \frac{\partial \xi^{(2)}}{\partial \xi} & 0 \\ \frac{\partial \xi w^{(2)}}{\partial \xi} & \frac{\partial \xi w^{(2)}}{\partial w} \\ 0 & \frac{\partial w^{(2)}}{\partial w} \end{bmatrix} \begin{bmatrix} F_\xi & 0 \\ 0 & F_w \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix} \end{aligned} \tag{A.63}$$

$$= \begin{bmatrix} \frac{\partial \xi^{(2)}}{\partial \xi} F_\xi \xi \\ \frac{\partial \xi w^{(2)}}{\partial \xi} F_\xi \xi + \frac{\partial \xi w^{(2)}}{\partial w} F_w w \\ \frac{\partial w^{(2)}}{\partial w} F_w w \end{bmatrix}$$

$$= \begin{bmatrix} D_\xi & 0 & 0 \\ 0 & D_{u_m} & 0 \\ 0 & 0 & D_w \end{bmatrix} \begin{bmatrix} \xi^{(2)} \\ \xi w^{(2)} \\ w^{(2)} \end{bmatrix}$$

which is the expected result. $\triangleleft$

a. Calculation of D_w

The calculation of the structural matrix D_w is reasonably straightforward for any specific system having a given matrix F_w. However, it is complicated in the general case by the fact that, unlike the matrix A, F_w does not have enough of a specific form in general to lend itself to iterative calculation. Instead, we will look at all the different possible cases for $w \in R^1$ and $w \in R^2$. Since the matrix F_w is assumed to be in block diagonal (Jordan) canonical form with eigenvalues having negative real parts, there are four possible cases:

- $F_w = [\lambda_1]$, with $\lambda_1 \neq 0$. That is, $w \in R^1$ and the matrix F_w has a single real eigenvalue, which is non-zero.
- $F_w = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, with $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. That is, $w \in R^2$ and the matrix F_w has two distinct real eigenvalues, which are both non-zero.
- $F_w = \begin{bmatrix} \lambda_1 & -\omega_w \\ \omega_w & \lambda_1 \end{bmatrix}$, with $\lambda_1 \neq 0$ and $\omega_w \neq 0$. That is, $w \in R^2$ and the matrix F_w has two complex conjugate eigenvalues, with non-zero real parts.
- $F_w = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$, with $\lambda_1 \neq 0$. That is, $w \in R^2$ and the matrix F_w has one real non-zero eigenvalue in a Jordan block.

Now we can look at theorems for these cases which give D_w.

Theorem 5.2 **(D_w 1D)** *For $w = [w_1] \in R^1$, and $F_w = [\lambda_1]$, with $\lambda_1 \neq 0$, the matrix D_w, defined by equation A.61, is given by the formula*

$$D_w = [2\lambda_1] \tag{A.64}$$

Proof. We prove this case by direct calculation. Taking the definition of D_w, equation A.61, and applying it to the one dimensional case, we get

$$\begin{aligned} D_w w_1^2 &\equiv \frac{\partial w_1^2}{\partial w_1} F_w w_1 \\ &= 2 w_1 \lambda_1 w_1 \\ &= [2\lambda_1]\, w_1^2 \end{aligned} \tag{A.65}$$

which is the expected result. ◁

Theorem 5.3 **(D_w 2D Distinct Real)** *For $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in R^2$, and $F_w = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, with $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, the matrix D_w, defined by equation A.61, is given by the formula*

$$D_w = \begin{bmatrix} 2\lambda_1 & 0 & 0 \\ 0 & (\lambda_1 + \lambda_2) & 0 \\ 0 & 0 & 2\lambda_2 \end{bmatrix} \tag{A.66}$$

Proof. We prove this case by direct calculation. Taking the definition of D_w, equation A.61, and applying it to the applicable two dimensional case, we get

$$\begin{aligned} D_w \begin{bmatrix} w_1^2 \\ w_1 w_2 \\ w_2^2 \end{bmatrix} &\equiv \begin{bmatrix} \frac{\partial w_1^2}{\partial w_1} & 0 \\ \frac{\partial w_1 w_2}{\partial w_1} & \frac{\partial w_1 w_2}{\partial w_2} \\ 0 & \frac{\partial w_2^2}{\partial w_2} \end{bmatrix} F_w \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ &= \begin{bmatrix} 2w_1 & 0 \\ w_2 & w_1 \\ 0 & 2w_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ &= \begin{bmatrix} 2\lambda_1 w_1^2 \\ (\lambda_1 + \lambda_2)\, w_1 w_2 \\ 2\lambda_2 w_2^2 \end{bmatrix} \end{aligned} \tag{A.67}$$

$$= \begin{bmatrix} 2\lambda_1 & 0 & 0 \\ 0 & (\lambda_1 + \lambda_2) & 0 \\ 0 & 0 & 2\lambda_2 \end{bmatrix} \begin{bmatrix} w_1^2 \\ w_1 w_2 \\ w_2^2 \end{bmatrix}$$

which is the expected result. ◁

Theorem 5.4 (D_w 2D Complex Conjugates) *For* $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in R^2$, *and* $F_w = \begin{bmatrix} \lambda_1 & -\omega_w \\ \omega_w & \lambda_1 \end{bmatrix}$, *with* $\lambda_1 \neq 0$ *and* $\omega_w \neq 0$, *the matrix* D_w, *defined by equation A.61, is given by the formula*

$$D_w = \begin{bmatrix} 2\lambda_1 & -2\omega_w & 0 \\ \omega_w & 2\lambda_1 & -\omega_w \\ 0 & 2\omega_w & 2\lambda_1 \end{bmatrix} \tag{A.68}$$

Proof. We prove this case by direct calculation. Taking the definition of D_w, equation A.61, and applying it to the applicable two dimensional case, we get

$$D_w \begin{bmatrix} w_1^2 \\ w_1 w_2 \\ w_2^2 \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial w_1^2}{\partial w_1} & 0 \\ \frac{\partial w_1 w_2}{\partial w_1} & \frac{\partial w_1 w_2}{\partial w_2} \\ 0 & \frac{\partial w_2^2}{\partial w_2} \end{bmatrix} F_w \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{A.69}$$

$$= \begin{bmatrix} 2w_1 & 0 \\ w_2 & w_1 \\ 0 & 2w_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & -\omega_w \\ \omega_w & \lambda_1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\lambda_1 w_1^2 - 2\omega_w w_1 w_2 \\ \omega_w w_1^2 + 2\lambda_1 w_1 w_2 - \omega_w w_2^2 \\ 2\omega_w w_1 w_2 + 2\lambda_1 w_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\lambda_1 & -2\omega_w & 0 \\ \omega_w & 2\lambda_1 & -\omega_w \\ 0 & 2\omega_w & 2\lambda_1 \end{bmatrix} \begin{bmatrix} w_1^2 \\ w_1 w_2 \\ w_2^2 \end{bmatrix}$$

which is the expected result. ◁

Theorem 5.5 **(D_w 2D Jordan Block)** *For* $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in R^2$*, and* $F_w = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$*, with* $\lambda_1 \neq 0$*, the matrix* D_w*, defined by equation A.61, is given by the formula*

$$D_w = \begin{bmatrix} 2\lambda_1 & 2 & 0 \\ 0 & 2\lambda_1 & 1 \\ 0 & 0 & 2\lambda_1 \end{bmatrix} \tag{A.70}$$

Proof. We prove this case by direct calculation. Taking the definition of D_w, equation A.61, and applying it to the applicable two dimensional case, we get

$$D_w \begin{bmatrix} w_1^2 \\ w_1 w_2 \\ w_2^2 \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial w_1^2}{\partial w_1} & 0 \\ \frac{\partial w_1 w_2}{\partial w_1} & \frac{\partial w_1 w_2}{\partial w_2} \\ 0 & \frac{\partial w_2^2}{\partial w_2} \end{bmatrix} F_w \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{A.71}$$

$$= \begin{bmatrix} 2w_1 & 0 \\ w_2 & w_1 \\ 0 & 2w_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\lambda_1 w_1^2 + 2w_1 w_2 \\ 2\lambda_1 w_1 w_2 + w_2^2 \\ 2\lambda_1 w_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\lambda_1 & 2 & 0 \\ 0 & 2\lambda_1 & 1 \\ 0 & 0 & 2\lambda_1 \end{bmatrix} \begin{bmatrix} w_1^2 \\ w_1 w_2 \\ w_2^2 \end{bmatrix}$$

which is the expected result. ◁

b. Calculation of D_{u_m}

The calculation of the structural matrix D_{u_m} is reasonably straightforward for the general case. We show it in a theorem.

Theorem 5.6 **(General Structure of D_{u_m})** *For any vector of linearly uncontrollable states* $\sigma \in R^s$*, such that* $\sigma = \begin{bmatrix} \xi \\ w \end{bmatrix}$*, with* $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_t \end{bmatrix} \in R^t$ *and* $w = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} \in$

R^m, and with linearly uncontrollable state coefficient matrix $F_\sigma \in R^{s\times s}$ of the block form

$$F_\sigma = \begin{bmatrix} F_\xi & 0 \\ 0 & F_w \end{bmatrix} \tag{A.72}$$

where $F_\xi \in R^{t\times t}$ and $F_w \in R^{m\times m}$, the structural matrix $D_{u_m} \in R^{tm\times tm}$, defined by the relation

$$D_{u_m}\xi w^{(2)} \equiv \frac{\partial \xi w^{(2)}}{\partial \xi} F_\xi \xi + \frac{\partial \xi w^{(2)}}{\partial w} F_w w \tag{A.73}$$

is given by the block formula

$$D_{u_m} = \begin{bmatrix} F_\xi & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & F_\xi \end{bmatrix} + \begin{bmatrix} F_{w_{11}} I & \cdots & F_{w_{1m}} I \\ \vdots & & \vdots \\ F_{w_{m1}} I & \cdots & F_{w_{mm}} I \end{bmatrix} \tag{A.74}$$

where the scalar values $F_{w_{ij}}$ are the entries in the matrix F_w, and $I \in R^{t\times t}$ is the identity matrix.

Proof. We prove the general case by exploiting the structure of $\xi w^{(2)}$. We have

$$\xi w^{(2)} = \begin{bmatrix} \xi w_1 \\ \vdots \\ \xi w_m \end{bmatrix} \tag{A.75}$$

which allows us to calculate

$$\frac{\partial \xi w^{(2)}}{\partial \xi} F_\xi \xi = \frac{\partial \begin{bmatrix} \xi w_1 \\ \vdots \\ \xi w_m \end{bmatrix}}{\partial \xi} F_\xi \xi = \begin{bmatrix} w_1 I \\ \vdots \\ w_m I \end{bmatrix} F_\xi \xi = \begin{bmatrix} F_\xi \xi w_1 \\ \vdots \\ F_\xi \xi w_m \end{bmatrix} \tag{A.76}$$

and

$$\frac{\partial \xi w^{(2)}}{\partial w} F_w w = \frac{\partial \begin{bmatrix} \xi w_1 \\ \vdots \\ \xi w_m \end{bmatrix}}{\partial w} F_w w \tag{A.77}$$

$$= \begin{bmatrix} \xi & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \xi \end{bmatrix} \begin{bmatrix} F_{w_{11}} w_1 + \ldots + F_{w_{1m}} w_m \\ \vdots \\ F_{w_{m1}} w_1 + \ldots + F_{w_{mm}} w_m \end{bmatrix}$$

$$= \begin{bmatrix} \xi \left(F_{w_{11}} w_1 + \ldots + F_{w_{1m}} w_m \right) \\ \vdots \\ \xi \left(F_{w_{m1}} w_1 + \ldots + F_{w_{mm}} w_m \right) \end{bmatrix}$$

We put these together to get

$$\begin{aligned} D_{u_m} \xi w^{(2)} &\equiv \frac{\partial \xi w^{(2)}}{\partial \xi} F_\xi \xi + \frac{\partial \xi w^{(2)}}{\partial w} F_w w \\ &= \begin{bmatrix} F_\xi \xi w_1 \\ \vdots \\ F_\xi \xi w_m \end{bmatrix} + \begin{bmatrix} \xi \left(F_{w_{11}} w_1 + \ldots + F_{w_{1m}} w_m \right) \\ \vdots \\ \xi \left(F_{w_{m1}} w_1 + \ldots + F_{w_{mm}} w_m \right) \end{bmatrix} \\ &= \left(\begin{bmatrix} F_\xi & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & F_\xi \end{bmatrix} + \begin{bmatrix} F_{w_{11}} I & \cdots & F_{w_{1m}} I \\ \vdots & & \vdots \\ F_{w_{m1}} I & \cdots & F_{w_{mm}} I \end{bmatrix} \right) \begin{bmatrix} \xi w_1 \\ \vdots \\ \xi w_m \end{bmatrix} \end{aligned} \tag{A.78}$$

which is the expected result. ◃

c. Calculation of D_ξ

Now we come to the calculation of the structural matrix D_ξ, which is used in calculation of the Poincare normal form for the various types of bifurcations, and in the control of the center manifold for the stabilization of linearly unstabilizable bifurcations. In this section we will calculate the general structure of the matrix D_ξ, and in subsequent sections we will calculate the specific values for the various types of bifurcations. We start with a theorem on the general structure.

Theorem 5.7 (General Structure of D_ξ) *For any vector of center states $\xi \in R^t$, such that $\xi = \begin{bmatrix} \mu \\ z \end{bmatrix}$, with $\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_r \end{bmatrix} \in R^r$ and $z = \begin{bmatrix} z_1 \\ \vdots \\ z_q \end{bmatrix} \in R^q$, and with center state coefficient matrix $F_\xi \in R^{t \times t}$ of the block form*

$$F_\xi = \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} \tag{A.79}$$

the structural matrix D_ξ, defined by the relation

$$D_\xi \xi^{(2)} \equiv \frac{\partial \xi^{(2)}}{\partial \xi} F_\xi \xi \tag{A.80}$$

has the block form

$$D_\xi = \begin{bmatrix} 0 & 0 & 0 \\ D_\rho & D_{\mu z} & 0 \\ 0 & D_\eta & D_{zz} \end{bmatrix} \tag{A.81}$$

where the sub-matrices D_ρ, $D_{\mu z}$, D_η and D_{zz} are defined by the relations

$$D_\rho \mu^{(2)} \equiv \frac{\partial \mu z^{(2)}}{\partial z} F_\mu \mu \tag{A.82}$$

$$D_{\mu z} \mu z^{(2)} \equiv \frac{\partial \mu z^{(2)}}{\partial z} F_z z \tag{A.83}$$

$$D_\eta \mu z^{(2)} \equiv \frac{\partial z^{(2)}}{\partial z} F_\mu \mu \tag{A.84}$$

$$D_{zz} z^{(2)} \equiv \frac{\partial z^{(2)}}{\partial z} F_z z \tag{A.85}$$

Proof. Since $\xi = \begin{bmatrix} \mu \\ z \end{bmatrix}$, the quadratic state vector has the form

$$\xi^{(2)} = \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} \tag{A.86}$$

So

$$D_\xi \xi^{(2)} \equiv \frac{\partial \xi^{(2)}}{\partial \xi} F_\xi \xi \tag{A.87}$$

$$= \begin{bmatrix} \frac{\partial \mu^{(2)}}{\partial \mu} & 0 \\ \frac{\partial \mu z^{(2)}}{\partial \mu} & \frac{\partial \mu z^{(2)}}{\partial z} \\ 0 & \frac{\partial z^{(2)}}{\partial z} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} \begin{bmatrix} \mu \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \frac{\partial \mu z^{(2)}}{\partial z} F_\mu \mu + \frac{\partial \mu z^{(2)}}{\partial z} F_z z \\ \frac{\partial z^{(2)}}{\partial z} F_\mu \mu + \frac{\partial z^{(2)}}{\partial z} F_z z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ D_\rho & D_{\mu z} & 0 \\ 0 & D_\eta & D_{zz} \end{bmatrix} \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix}$$

which is the expected result. ◁

i. Calculation of $D_{\mu z}$

The calculation of the structural matrix $D_{\mu z}$ is reasonably straightforward for the general case, so we show it in a theorem.

Theorem 5.8 (The General Structure of $D_{\mu z}$) *For any vector of center states $\xi \in R^t$, such that $\xi = \begin{bmatrix} \mu \\ z \end{bmatrix}$, with $\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_r \end{bmatrix} \in R^r$ and $z = \begin{bmatrix} z_1 \\ \vdots \\ z_q \end{bmatrix} \in R^q$, and with center state coefficient matrix $F_\xi \in R^{t \times t}$ of the block form*

$$F_\xi = \begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix} \tag{A.88}$$

the structural matrix $D_{\mu z} \in R^{rq \times rq}$, defined by the relation

$$D_{\mu z} \mu z^{(2)} \equiv \frac{\partial \mu z^{(2)}}{\partial z} F_z z \tag{A.89}$$

has the block form

$$D_{\mu z} = \begin{bmatrix} F_{z_{11}} I & \cdots & F_{z_{1q}} I \\ \vdots & & \vdots \\ F_{z_{q1}} I & \cdots & F_{z_{qq}} I \end{bmatrix} \tag{A.90}$$

where $F_{z_{ij}}$ are the scalar elements of the matrix F_z, and $I \in R^{r \times r}$ is the identity matrix.

Proof. We prove the general case by exploiting the structure of $\mu z^{(2)}$. We have

$$\mu z^{(2)} = \begin{bmatrix} \mu z_1 \\ \vdots \\ \mu z_q \end{bmatrix} \tag{A.91}$$

which allows us to calculate

$$\begin{aligned}
D_{\mu z}\mu z^{(2)} \equiv \frac{\partial \mu z^{(2)}}{\partial z} F_z z &= \frac{\partial \begin{bmatrix} \mu z_1 \\ \vdots \\ \mu z_q \end{bmatrix}}{\partial z} F_z z \\
&= \begin{bmatrix} \mu & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mu \end{bmatrix} \begin{bmatrix} F_{z_{11}} z_1 + \ldots + F_{z_{1q}} z_q \\ \vdots \\ F_{z_{q1}} z_1 + \ldots + F_{z_{qq}} z_q \end{bmatrix} \\
&= \begin{bmatrix} \mu \left(F_{z_{11}} z_1 + \ldots + F_{z_{1q}} z_q \right) \\ \vdots \\ \mu \left(F_{z_{q1}} z_1 + \ldots + F_{z_{qq}} z_q \right) \end{bmatrix} \\
&= \begin{bmatrix} F_{z_{11}} I & \cdots & F_{z_{1q}} I \\ \vdots & & \vdots \\ F_{z_{q1}} I & \cdots & F_{z_{qq}} I \end{bmatrix} \begin{bmatrix} \mu z_1 \\ \vdots \\ \mu z_q \end{bmatrix}
\end{aligned} \tag{A.92}$$

which is the expected result. ◁

6. CALCULATING D_ξ FOR SPECIFIC BIFURCATIONS

In this section we will calculate the specific forms of the structural matrix D_ξ for individual types of bifurcations on the center manifold. To calculate the matrix D_ξ, we need to calculate the four sub-matrices D_ρ, $D_{\mu z}$, D_η and D_{zz}, and then substitute them into equation A.81.

a. One Dimensional Bifurcations: Saddle-Node, Transcritical and Pitchfork

Saddle-node, transcritical and pitchfork bifurcations are one dimensional co-dimension one bifurcations characterized by the matrices $F_\mu = [F_{\mu_1}]$ and $F_z = [0]$. For a saddle node bifurcation $F_{\mu_1} \neq 0$. For a transcritical or pitchfork bifurcation, $F_{\mu_1} = 0$. Again, we present our result in the form of a theorem.

Theorem 6.1 **(D_ξ for a 1D Bifurcation)** *For a one dimensional bifurcation characterized by the matrices $F_\mu = [F_{\mu_1}]$ and $F_z = [0]$, the structural matrix D_ξ is given by*

$$D_\xi = \begin{bmatrix} 0 & 0 & 0 \\ F_{\mu_1} & 0 & 0 \\ 0 & 2F_{\mu_1} & 0 \end{bmatrix} \tag{A.93}$$

with

$$D_\rho = [F_{\mu_1}] \tag{A.94}$$
$$D_{\mu z} = [0] \tag{A.95}$$
$$D_\eta = [2F_{\mu_1}] \tag{A.96}$$
$$D_{zz} = [0] \tag{A.97}$$

Proof. Since $F_z = 0$, we instantly have $D_{\mu z} = 0$ and $D_{zz} = 0$ by equations A.83 and A.85, since both matrices are linearly dependent on F_z. Since $\mu = [\mu_1]$ and $z = [z_1]$, we have $\mu^{(2)} = [\mu_1^2]$, $\mu z^{(2)} = [\mu z_1]$ and $z^{(2)} = [z_1^2]$. Evaluating equations A.82 and A.84 for $F_\mu = [F_{\mu_1}]$ we get $D_\rho = [F_{\mu_1}]$ and $D_\eta = [2F_{\mu_1}]$. Plugging into equation A.81 gives the expected result. ◃

b. Hopf Bifurcations

Hopf bifurcations are two dimensional co-dimension one bifurcations characterized by the matrices $F_\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $F_z = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix}$, where $\omega_0 \neq 0$. We present our result in the form of a theorem.

Theorem 6.2 **(D_ξ for Hopf Bifurcations)** *For a Hopf bifurcation characterized by the matrices $F_\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $F_z = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix}$, where $\omega_0 \neq 0$, the structural*

matrix D_ξ is given by

$$D_\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_0 & 0 & 0 & 0 \\ 0 & \omega_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\omega_0 & 0 \\ 0 & 0 & 0 & \omega_0 & 0 & -\omega_0 \\ 0 & 0 & 0 & 0 & 2\omega_0 & 0 \end{bmatrix} \tag{A.98}$$

with

$$D_\rho = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A.99}$$

$$D_{\mu z} = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \tag{A.100}$$

$$D_\eta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{A.101}$$

$$D_{zz} = \begin{bmatrix} 0 & -2\omega_0 & 0 \\ \omega_0 & 0 & -\omega_0 \\ 0 & 2\omega_0 & 0 \end{bmatrix} \tag{A.102}$$

Proof. Since $F_\mu = 0$, we instantly have $D_\rho = 0$ and $D_\eta = 0$ by equations A.82 and A.84, since both matrices are linearly dependent on F_μ. Since $\mu = [\mu_1]$ and $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, we have $\mu^{(2)} = [\mu_1^2]$, $\mu z^{(2)} = \begin{bmatrix} \mu_1 z_1 \\ \mu_1 z_2 \end{bmatrix}$ and $z^{(2)} = \begin{bmatrix} z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix}$. Using the "General Structure of $D_{\mu z}$" theorem, with $F_z = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix}$, we get

$$D_{\mu z} = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \tag{A.103}$$

Evaluating equation A.85 for D_{zz}, we get

$$D_{zz} \begin{bmatrix} z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix} = \begin{bmatrix} \frac{\partial z_1^2}{\partial z_1} & 0 \\ \frac{\partial z_1 z_2}{\partial z_1} & \frac{\partial z_1 z_2}{\partial z_2} \\ 0 & \frac{\partial z_2^2}{\partial z_2} \end{bmatrix} \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{A.104}$$

$$= \begin{bmatrix} 2z_1 & 0 \\ z_2 & z_1 \\ 0 & 2z_2 \end{bmatrix} \begin{bmatrix} -\omega_0 z_2 \\ \omega_0 z_1 \end{bmatrix} = \begin{bmatrix} -2\omega_0 z_1 z_2 \\ \omega_0 z_1^2 - \omega_0 z_2^2 \\ 2\omega_0 z_1 z_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2\omega_0 & 0 \\ \omega_0 & 0 & -\omega_0 \\ 0 & 2\omega_0 & 0 \end{bmatrix} \begin{bmatrix} z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix}$$

Plugging into equation A.81 gives the expected result. ◁

c. Double Zero Bifurcations

Co-dimension one double zero bifurcations are characterized by the matrices $F_\mu = \begin{bmatrix} 0 \\ F_{\mu_2} \end{bmatrix}$ and $F_z = \begin{bmatrix} 0 & \lambda_0 \\ 0 & 0 \end{bmatrix}$, where $\lambda_0 \neq 0$. We present our result in the form of a theorem.

Theorem 6.3 (**D_ξ for Double Zero Bifurcatio**) *For a double zero bifurcation characterized by the matrices $F_\mu = \begin{bmatrix} 0 \\ F_{\mu_2} \end{bmatrix}$ and $F_z = \begin{bmatrix} 0 & \lambda_0 \\ 0 & 0 \end{bmatrix}$, where $\lambda_0 \neq 0$, the structural matrix D_ξ is given by*

$$D_\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_0 & 0 & 0 & 0 \\ F_{\mu_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\lambda_0 & 0 \\ 0 & F_{\mu_2} & 0 & 0 & 0 & \lambda_0 \\ 0 & 0 & 2F_{\mu_2} & 0 & 0 & 0 \end{bmatrix} \tag{A.105}$$

with

$$D_\rho = \begin{bmatrix} 0 \\ F_{\mu_2} \end{bmatrix} \tag{A.106}$$

$$D_{\mu z} = \begin{bmatrix} 0 & \lambda_0 \\ 0 & 0 \end{bmatrix} \tag{A.107}$$

$$D_\eta = \begin{bmatrix} 0 & 0 \\ F_{\mu_2} & 0 \\ 0 & 2F_{\mu_2} \end{bmatrix} \tag{A.108}$$

$$D_{zz} = \begin{bmatrix} 0 & 2\lambda_0 & 0 \\ 0 & 0 & \lambda_0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A.109}$$

Proof. Since $\mu = [\mu_1]$ and $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, we have $\mu^{(2)} = [\mu_1^2]$, $\mu z^{(2)} = \begin{bmatrix} \mu_1 z_1 \\ \mu_1 z_2 \end{bmatrix}$ and $z^{(2)} = \begin{bmatrix} z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix}$. Using the "General Structure of $D_{\mu z}$" theorem, with $F_z = \begin{bmatrix} 0 & \lambda_0 \\ 0 & 0 \end{bmatrix}$, we get

$$D_{\mu z} = \begin{bmatrix} 0 & \lambda_0 \\ 0 & 0 \end{bmatrix} \tag{A.110}$$

Evaluating equations A.82, A.84 and A.85 for D_ρ, D_η and D_{zz}, we get

$$\begin{aligned} D_\rho \left[\mu_1^2\right] &= \begin{bmatrix} \frac{\partial \mu_1 z_1}{\partial z_1} & \frac{\partial \mu_1 z_1}{\partial z_2} \\ \frac{\partial \mu_1 z_2}{\partial z_1} & \frac{\partial \mu_1 z_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} 0 \\ F_{\mu_2} \end{bmatrix} \mu_1 \\ &= \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{bmatrix} \begin{bmatrix} 0 \\ F_{\mu_2}\mu_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ F_{\mu_2} \end{bmatrix} \left[\mu_1^2\right] \end{aligned} \tag{A.111}$$

and

$$\begin{aligned} D_\eta \begin{bmatrix} \mu_1 z_1 \\ \mu_1 z_2 \end{bmatrix} &= \begin{bmatrix} \frac{\partial z_1^2}{\partial z_1} & 0 \\ \frac{\partial z_1 z_2}{\partial z_1} & \frac{\partial z_1 z_2}{\partial z_2} \\ 0 & \frac{\partial z_2^2}{\partial z_2} \end{bmatrix} \begin{bmatrix} 0 \\ F_{\mu_2} \end{bmatrix} \mu_1 \\ &= \begin{bmatrix} 2z_1 & 0 \\ z_2 & z_1 \\ 0 & 2z_2 \end{bmatrix} \begin{bmatrix} 0 \\ F_{\mu_2}\mu_1 \end{bmatrix} = \begin{bmatrix} 0 \\ F_{\mu_2}\mu_1 z_1 \\ 2F_{\mu_2}\mu_1 z_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ F_{\mu_2} & 0 \\ 0 & 2F_{\mu_2} \end{bmatrix} \begin{bmatrix} \mu_1 z_1 \\ \mu_1 z_2 \end{bmatrix} \end{aligned} \tag{A.112}$$

and

$$D_{zz}\begin{bmatrix} z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix} = \begin{bmatrix} \frac{\partial z_1^2}{\partial z_1} & 0 \\ \frac{\partial z_1 z_2}{\partial z_1} & \frac{\partial z_1 z_2}{\partial z_2} \\ 0 & \frac{\partial z_2^2}{\partial z_2} \end{bmatrix} \begin{bmatrix} 0 & \lambda_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{A.113}$$

$$= \begin{bmatrix} 2z_1 & 0 \\ z_2 & z_1 \\ 0 & 2z_2 \end{bmatrix} \begin{bmatrix} \lambda_0 z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\lambda_0 z_1 z_2 \\ \lambda_0 z_2^2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2\lambda_0 & 0 \\ 0 & 0 & \lambda_0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix}$$

Plugging into equation A.81 gives the expected result. $\triangleleft$

d. Two Zeroes Bifurcations

Co-dimension one two zeroes bifurcations are characterized by the matrices $F_\mu = \begin{bmatrix} F_{\mu_1} \\ F_{\mu_2} \end{bmatrix}$ and $F_z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. We present our result in the form of a theorem.

Theorem 6.4 (D_ξ for Two Zeroes Bifurcation) *For a two zeroes bifurcation characterized by the matrices* $F_\mu = \begin{bmatrix} F_{\mu_1} \\ F_{\mu_2} \end{bmatrix}$ *and* $F_z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, *the structural matrix* D_ξ *is given by*

$$D_\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ F_{\mu_1} & 0 & 0 & 0 & 0 & 0 \\ F_{\mu_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2F_{\mu_1} & 0 & 0 & 0 & 0 \\ 0 & F_{\mu_2} & F_{\mu_1} & 0 & 0 & 0 \\ 0 & 0 & 2F_{\mu_2} & 0 & 0 & 0 \end{bmatrix} \tag{A.114}$$

with

$$D_\rho = \begin{bmatrix} F_{\mu_1} \\ F_{\mu_2} \end{bmatrix} \tag{A.115}$$

$$D_{\mu z} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{A.116}$$

$$D_\eta = \begin{bmatrix} 2F_{\mu_1} & 0 \\ F_{\mu_2} & F_{\mu_1} \\ 0 & 2F_{\mu_2} \end{bmatrix} \tag{A.117}$$

$$D_{zz} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A.118}$$

Proof. Since $F_z = 0$, we instantly have $D_{\mu z} = 0$ and $D_{zz} = 0$ by equations A.83 and A.85, since both matrices are linearly dependent on F_z. Since $\mu = [\mu_1]$ and $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, we have $\mu^{(2)} = [\mu_1^2]$, $\mu z^{(2)} = \begin{bmatrix} \mu_1 z_1 \\ \mu_1 z_2 \end{bmatrix}$ and $z^{(2)} = \begin{bmatrix} z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix}$. Evaluating equations A.82 and A.84 for D_ρ and D_η, we get

$$\begin{aligned} D_\rho \left[\mu_1^2\right] &= \begin{bmatrix} \frac{\partial \mu_1 z_1}{\partial z_1} & \frac{\partial \mu_1 z_1}{\partial z_2} \\ \frac{\partial \mu_1 z_2}{\partial z_1} & \frac{\partial \mu_1 z_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} F_{\mu_1} \\ F_{\mu_2} \end{bmatrix} \mu_1 \\ &= \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{bmatrix} \begin{bmatrix} F_{\mu_1}\mu_1 \\ F_{\mu_2}\mu_1 \end{bmatrix} \\ &= \begin{bmatrix} F_{\mu_1} \\ F_{\mu_2} \end{bmatrix} \left[\mu_1^2\right] \end{aligned} \tag{A.119}$$

and

$$\begin{aligned} D_\eta \begin{bmatrix} \mu_1 z_1 \\ \mu_1 z_2 \end{bmatrix} &= \begin{bmatrix} \frac{\partial z_1^2}{\partial z_1} & 0 \\ \frac{\partial z_1 z_2}{\partial z_1} & \frac{\partial z_1 z_2}{\partial z_2} \\ 0 & \frac{\partial z_2^2}{\partial z_2} \end{bmatrix} \begin{bmatrix} F_{\mu_1} \\ F_{\mu_2} \end{bmatrix} \mu_1 \\ &= \begin{bmatrix} 2z_1 & 0 \\ z_2 & z_1 \\ 0 & 2z_2 \end{bmatrix} \begin{bmatrix} F_{\mu_1}\mu_1 \\ F_{\mu_2}\mu_1 \end{bmatrix} = \begin{bmatrix} 2F_{\mu_1}\mu_1 z_1 \\ F_{\mu_2}\mu_1 z_1 + F_{\mu_1}\mu_1 z_2 \\ 2F_{\mu_2}\mu_1 z_2 \end{bmatrix} \\ &= \begin{bmatrix} 2F_{\mu_1} & 0 \\ F_{\mu_2} & F_{\mu_1} \\ 0 & 2F_{\mu_2} \end{bmatrix} \begin{bmatrix} \mu_1 z_1 \\ \mu_1 z_2 \end{bmatrix} \end{aligned} \tag{A.120}$$

Plugging into equation A.81 gives the expected result. ◁

APPENDIX B. QUADRATIC TRANSFORMATIONS

In this Appendix we will develop the quadratic coordinate transformations which implement the quadratic normal form developed in Chapter V. From the Separation Principle theorem and the "Constraints on H from g(x)" lemma of that chapter, we have to solve six matrix equations subject to the constraints imposed by four additional matrix equations. The matrix equations to solve are

$$H_{yc}D_c - AH_{yc} = \tilde{Q}_{yc} - \check{Q}_{yc} \tag{B.1}$$

$$H_{ym}D_m - AH_{ym} = \tilde{Q}_{ym} - \check{Q}_{ym} \tag{B.2}$$

$$H_{yu}D_\sigma - AH_{yu} = \tilde{Q}_{yu} - \check{Q}_{yu} \tag{B.3}$$

$$H_{\sigma c}D_c - F_\sigma H_{\sigma c} = \tilde{Q}_{\sigma c} - \check{Q}_{\sigma c} \tag{B.4}$$

$$H_{\sigma m}D_m - F_\sigma H_{\sigma m} = \tilde{Q}_{\sigma m} - \check{Q}_{\sigma m} \tag{B.5}$$

$$H_{\sigma u}D_\sigma - F_\sigma H_{\sigma u} = \tilde{Q}_{\sigma u} - \check{Q}_{\sigma u} \tag{B.6}$$

and the constraint equations are

$$\tilde{G}_{\sigma u} - H_{\sigma m}D_{B_\sigma} = 0 \tag{B.7}$$

$$\tilde{G}_{\sigma c} - H_{\sigma c}D_{B_y} = 0 \tag{B.8}$$

$$\tilde{G}_{yu} - H_{ym}D_{B_\sigma} = \check{G}_{yu} \tag{B.9}$$

$$\tilde{G}_{yc} - H_{yc}D_{B_y} = \check{G}_{yc} \tag{B.10}$$

Now we will look at actually solving these equations for the coordinate transformation matrices, the normal form matrices, and the components of the feedback.

1. CONTROLLABLE/UNCONTROLLABLE PART

The controllable/uncontrollable portion of the quadratic coordinate transformation process is given by the solutions H_{yu} and $\check{Q}_{yu}$ of equation B.3, rewritten here for convenience,

$$H_{yu}D_\sigma - AH_{yu} = \tilde{Q}_{yu} - \check{Q}_{yu} \tag{B.11}$$

which is not subject to any of the constraint equations. Because the quadratic normal form of this portion was proven to be zero in Chapter V after feedback was applied, the coefficient matrix $\check{Q}_{yu}$ can have at most a non-zero bottom row after the quadratic coordinate transformation is complete. So $\check{Q}_{yu}$ has the form

$$\check{Q}_{yu} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \check{Q}_{yu_p} \end{bmatrix} \tag{B.12}$$

Equation B.11 is in the proper form to apply the Unstacking Theorem of Chapter V, which yields

$$\begin{bmatrix} D_\sigma^T & -I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -I \\ 0 & \cdots & \cdots & 0 & D_\sigma^T \end{bmatrix} \begin{bmatrix} H_{yu_1}^T \\ \vdots \\ H_{yu_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{yu_1}^T \\ \vdots \\ \tilde{Q}_{yu_p}^T \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \check{Q}_{yu_p}^T \end{bmatrix} \tag{B.13}$$

The solution of equation B.13 is the subject of the following theorem.

Theorem 1.1 (Controllable/Uncontrollable Solution) *The controllable/uncontrollable portion of the quadratic coordinate transformation required to put a system into quadratic normal form is given by the solutions H_{yu} and $\check{Q}_{yu}$ of the block matrix*

equation

$$\begin{bmatrix} H^T_{yu_1} \\ \vdots \\ H^T_{yu_p} \\ \check{Q}^T_{yu_p} \end{bmatrix} = \begin{bmatrix} I & 0 & & \cdots & & 0 \\ D^T_\sigma & -I & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & \ddots & -I & 0 \\ 0 & \cdots & \cdots & 0 & D^T_\sigma & I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \tilde{Q}^T_{yu_1} \\ \vdots \\ \tilde{Q}^T_{yu_p} \end{bmatrix} \tag{B.14}$$

where

$$H_{yu} = \begin{bmatrix} H_{yu_1} \\ \vdots \\ H_{yu_p} \end{bmatrix} \tag{B.15}$$

and

$$\check{Q}_{yu} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \check{Q}_{yu_p} \end{bmatrix} \tag{B.16}$$

and where the matrix D_σ is obtained from Appendix A, and the block elements $\tilde{Q}_{yu_j}$ are the rows of the coefficient matrix $\tilde{Q}_{yu}$.

Proof. The block matrix equation B.13 has p block equations in $p+1$ block unknowns, and is therefore underdetermined. One of the block variables is therefore a free variable, which we can pick. We choose $H^T_{yu_1} = 0$, which is one additional block equation, which we append to equation B.13 and rearrange to show $\check{Q}^T_{yu_p}$ as a variable. The resulting coefficient matrix is invertible as shown in equation B.14, since it is lower triangular with non-zero elements on the main diagonal. ◃

2. CONTROLLABLE/MIXED PART

The controllable/mixed portion of the quadratic coordinate transformation process is given by the solutions H_{ym}, $\check{Q}_{ym}$ and $\check{G}_{yu}$ of equation B.2, subject to the constraints of equation B.9, which are rewritten here for convenience,

$$H_{ym}D_m - AH_{ym} = \tilde{Q}_{ym} - \check{Q}_{ym} \tag{B.17}$$

$$\tilde{G}_{yu} - H_{ym}D_{B_\sigma} = \check{G}_{yu} \tag{B.18}$$

Because the quadratic normal form of this portion was proven to be zero in Chapter V after feedback was applied, the coefficient matrices $\check{Q}_{ym}$ and $\check{G}_{yu}$ can each have at most a non-zero bottom row after the quadratic coordinate transformation is complete. So $\check{Q}_{ym}$ and $\check{G}_{yu}$ have the form

$$\check{Q}_{ym} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \check{Q}_{ym_p} \end{bmatrix} \tag{B.19}$$

and

$$\check{G}_{yu} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \check{Q}_{yu_p} \end{bmatrix} \tag{B.20}$$

Equations B.17 and B.18 are in the proper form to apply the Unstacking Theorem of Chapter V, which yields

$$\begin{bmatrix} D_m^T & -I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -I \\ 0 & \cdots & \cdots & 0 & D_m^T \end{bmatrix} \begin{bmatrix} H_{ym_1}^T \\ \vdots \\ H_{ym_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{ym_1}^T \\ \vdots \\ \tilde{Q}_{ym_p}^T \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \check{Q}_{ym_p}^T \end{bmatrix} \tag{B.21}$$

$$\begin{bmatrix} D_{B_\sigma}^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_{B_\sigma}^T \end{bmatrix} \begin{bmatrix} H_{ym_1}^T \\ \vdots \\ H_{ym_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{yu_1}^T \\ \vdots \\ \tilde{G}_{yu_p}^T \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \check{G}_{yu_p}^T \end{bmatrix} \tag{B.22}$$

The solution of equations B.21 and B.22 is the subject of the following theorem.

Theorem 2.1 (Controllable/Mixed Solution) *The controllable/mixed portion of the quadratic coordinate transformation required to put a system into quadratic normal form is given by the solutions H_{ym}, $\check{Q}_{ym}$ and $\check{G}_{ym}$ of the block matrix equations*

$$\check{G}_{yu_p}^T = 0 \tag{B.23}$$

and

$$\begin{bmatrix} H^T_{ym_1} \\ \vdots \\ H^T_{ym_p} \\ \check{Q}^T_{ym_p} \end{bmatrix} = \begin{bmatrix} D^T_m & -I & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & \ddots & -I & 0 \\ 0 & \cdots & \cdots & 0 & D^T_m & I \\ D^T_{B_\sigma} & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & D^T_{B_\sigma} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{Q}^T_{ym_1} \\ \vdots \\ \tilde{Q}^T_{ym_p} \\ \tilde{G}^T_{yu_1} \\ \vdots \\ \tilde{G}^T_{yu_p} \end{bmatrix} \tag{B.24}$$

where

$$H_{ym} = \begin{bmatrix} H_{ym_1} \\ \vdots \\ H_{ym_p} \end{bmatrix} \tag{B.25}$$

and

$$\check{Q}_{ym} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \check{Q}_{ym_p} \end{bmatrix} \tag{B.26}$$

and

$$\check{G}_{yu} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \check{G}_{yu_p} \end{bmatrix} = 0 \tag{B.27}$$

and where the matrices D_m and D_{B_σ} are obtained from Appendix A, and the block elements $\tilde{Q}_{ym_j}$ and $\tilde{G}_{yu_j}$ are the rows of the coefficient matrices $\tilde{Q}_{ym}$ and $\tilde{G}_{yu}$, respectively.

Proof. The combined block matrix equations B.21 and B.22 have $sp(p+1)$ block equations in $sp(p+1)+s$ block unknowns, and are therefore underdetermined. One of the block variables is therefore a free variable, which we can pick. We choose $\check{G}_{yu_p} = 0$, which we show as a separate block matrix equation. Then we append equations B.21 and B.22 together and rearrange to show $\check{Q}^T_{ym_p}$ as a variable. The

resulting block matrix equation is

$$\begin{bmatrix} D_m^T & -I & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & \ddots & -I & 0 \\ 0 & \cdots & \cdots & 0 & D_m^T & I \\ D_{B_\sigma}^T & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & D_{B_\sigma}^T & 0 \end{bmatrix} \begin{bmatrix} H_{ym_1}^T \\ \vdots \\ H_{ym_p}^T \\ \check{Q}_{ym_p}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{ym_1}^T \\ \vdots \\ \tilde{Q}_{ym_p}^T \\ \tilde{G}_{yu_1}^T \\ \vdots \\ \tilde{G}_{yu_p}^T \end{bmatrix} \tag{B.28}$$

and it only remains to be shown that the coefficient matrix in equation B.28 is invertible. The proof of invertibility uses block Gaussian elimination to put the matrix into upper triangular form, and follows exactly the proof given in the Controllable/Mixed theorem in Chapter V, which will not be repeated here. ◁

3. CONTROLLABLE/CONTROLLABLE PART

The controllable/controllable portion of the quadratic coordinate transformation process is given by the solutions H_{yc}, $\check{Q}_{yc}$ and $\check{G}_{yc}$ of equation B.1, subject to the constraints of equation B.10, which are rewritten here for convenience

$$H_{yc}D_c - AH_{yc} = \tilde{Q}_{yc} - \check{Q}_{yc} \tag{B.29}$$

$$\tilde{G}_{yc} - H_{yc}D_{B_y} = \check{G}_{yc} \tag{B.30}$$

The quadratic normal form of this portion was proven in Chapter V to include only selected y_j^2 terms, which are $Q_{y_c}\begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix}$, with Q_{y_c} having the form

$$Q_{y_c} = \begin{bmatrix} 0 & 0 & \gamma_{13} & \cdots & \gamma_{1p} \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \gamma_{(p-2)p} \\ & & & \ddots & 0 \\ 0 & & \cdots & & 0 \end{bmatrix} \tag{B.31}$$

which is seen to be upper triangular with zeros on the main diagonal and first super diagonal. Now, the matrix $\check{Q}_{yc}$ is defined by the relation

$$\check{Q}_{yc}y^{(2)} = Q_{y_c}\begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \check{Q}_{yc_p} \end{bmatrix} y^{(2)} \tag{B.32}$$

Because of the structure inherent in the matrix Q_{y_c}, the controllable/controllable portion of the quadratic coordinate transformation is less amenable to a general solution than the other portions. Instead, the solution should be handled on a case by case basis.

4. UNCONTROLLABLE/CONTROLLABLE PART

The uncontrollable/controllable portion of the quadratic coordinate transformation process is given by the solutions $H_{\sigma c}$ and $\check{Q}_{\sigma c}$ of equation B.4, subject to the constraints of equation B.8, which are rewritten here for convenience,

$$H_{\sigma c}D_c - F_\sigma H_{\sigma c} = \tilde{Q}_{\sigma c} - \check{Q}_{\sigma c} \tag{B.33}$$

$$\tilde{G}_{\sigma c} - H_{\sigma c}D_{B_y} = 0 \tag{B.34}$$

The quadratic normal form of this portion was proven to include all of the y_j^2 terms and nothing else in Chapter V. If we define the y_j^2 terms as $Q_{\sigma_c}\begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix}$ then the matrix $\check{Q}_{\sigma c}$ has the form

$$\check{Q}_{\sigma c} y^{(2)} = Q_{\sigma_c}\begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \tag{B.35}$$

where the matrix $Q_{\sigma_c} \in R^{s \times p}$ has the block form

$$Q_{\sigma_c} = \begin{bmatrix} 0 \\ Q_{z_c} \\ Q_{w_c} \end{bmatrix} \tag{B.36}$$

In order to perform the matrix operations we need to define a new matrix which relates $\check{Q}_{\sigma c}$ to Q_{σ_c}. We do this in a lemma.

Lemma 4.1 (Definition of D_{q_c}) *The matrix $D_{q_c} \in R^{p \times \frac{p(p+1)}{2}}$ relates the rows of the matrix $\check{Q}_{\sigma c}$ to the rows of the matrix Q_{σ_c} as follows*

$$\check{Q}_{\sigma c_j} = Q_{\sigma_{c_j}} D_{q_c} \tag{B.37}$$

The matrix D_{q_c} is defined iteratively by the relations

$$D_{q_{c_1}} = [1] \tag{B.38}$$

$$D_{q_{c_{p+1}}} = \begin{bmatrix} D_{q_{c_p}} & 0_p & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{B.39}$$

where we have used the definition $0_p \in R^{p \times p} = 0$.

Proof. By the "Notation for Multi-Variable Taylor Series Expansions" lemma in Chapter III, the elements of the quadratic state vector $y^{(2)}$ are ordered by the rule

$y_h y_i > y_j y_k$ if $i > k$, or $h > j$ if $i = k$, which gives

$$y^{(2)} = \begin{bmatrix} y_1^2 \\ y_1 y_2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix} \tag{B.40}$$

The matrix D_{q_c} is the matrix which selects only the y_j^2 terms from this vector. ◁

So now we can state the form of the matrix $\check{Q}_{\sigma c}$, which is

$$\check{Q}_{\sigma c} = \begin{bmatrix} \check{Q}_{\sigma c_1} \\ \vdots \\ \check{Q}_{\sigma c_s} \end{bmatrix} = \begin{bmatrix} Q_{\sigma_{c_1}} D_{q_c} \\ \vdots \\ Q_{\sigma_{c_s}} D_{q_c} \end{bmatrix} \tag{B.41}$$

Applying the Unstacking Theorem from Chapter V to equations B.33 and B.34, we get

$$\left(\begin{bmatrix} D_c^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_c^T \end{bmatrix} - \begin{bmatrix} F_{\sigma_{11}} I & \cdots & F_{\sigma_{1s}} I \\ \vdots & & \vdots \\ F_{\sigma_{s1}} I & \cdots & F_{\sigma_{ss}} I \end{bmatrix} \right) \begin{bmatrix} H_{\sigma c_1}^T \\ \vdots \\ H_{\sigma c_s}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{\sigma c_1}^T \\ \vdots \\ \tilde{Q}_{\sigma c_s}^T \end{bmatrix} - \begin{bmatrix} \check{Q}_{\sigma c_1} \\ \vdots \\ \check{Q}_{\sigma c_s} \end{bmatrix} \tag{B.42}$$

$$\begin{bmatrix} D_{B_y}^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_{B_y}^T \end{bmatrix} \begin{bmatrix} H_{\sigma c_1}^T \\ \vdots \\ H_{\sigma c_s}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{\sigma c_1}^T \\ \vdots \\ \tilde{G}_{\sigma c_s}^T \end{bmatrix} \tag{B.43}$$

The solution to these equations is the subject of the next theorem.

Theorem 4.2 (Uncontrollable/Controllable Solution) *The uncontrollable/controllable portion of the quadratic coordinate transformation required to put a system into quadratic normal form is given by the solutions* $H_{\sigma c}$ *and* Q_{σ_c} *of the block matrix*

equation

$$
\begin{bmatrix} H^T_{\sigma c_1} \\ \vdots \\ H^T_{\sigma c_s} \\ Q^T_{\sigma c_1} \\ \vdots \\ Q^T_{\sigma c_s} \end{bmatrix} = \begin{bmatrix}
\left(D_c^T - F_{\sigma_{11}} I\right) & -F_{\sigma_{12}} I & \cdots & -F_{\sigma_{1s}} I & D^T_{qc} & 0 & \cdots & 0 \\
-F_{\sigma_{21}} I & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & -F_{\sigma_{(s-1)s}} I & \vdots & \ddots & \ddots & 0 \\
-F_{\sigma_{s1}} I & \cdots & -F_{\sigma_{s(s-1)}} I & \left(D_c^T - F_{\sigma_{ss}} I\right) & 0 & \cdots & 0 & D^T_{qc} \\
D^T_{By} & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & \vdots & & & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & & & \vdots \\
0 & \cdots & 0 & D^T_{By} & 0 & \cdots & \cdots & 0
\end{bmatrix}^{-1} \begin{bmatrix} \tilde{Q}^T_{\sigma c_1} \\ \vdots \\ \tilde{Q}^T_{\sigma c_s} \\ \tilde{G}^T_{\sigma c_1} \\ \vdots \\ \tilde{G}^T_{\sigma c_s} \end{bmatrix} \tag{B.44}
$$

Proof. We begin by plugging equation B.41 into equation B.42, and appending equations B.42 and B.43 together. The resulting block matrix equation is

$$
\begin{bmatrix}
\left(D_c^T - F_{\sigma_{11}} I\right) & -F_{\sigma_{12}} I & \cdots & -F_{\sigma_{1s}} I \\
-F_{\sigma_{21}} I & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & -F_{\sigma_{(s-1)s}} I \\
-F_{\sigma_{s1}} I & \cdots & -F_{\sigma_{s(s-1)}} I & \left(D_c^T - F_{\sigma_{ss}} I\right) \\
D^T_{By} & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & D^T_{By}
\end{bmatrix} \begin{bmatrix} H^T_{\sigma c_1} \\ \vdots \\ H^T_{\sigma c_s} \end{bmatrix} = \begin{bmatrix} \tilde{Q}^T_{\sigma c_1} - D^T_{qc} Q^T_{\sigma c_1} \\ \vdots \\ \tilde{Q}^T_{\sigma c_s} - D^T_{qc} Q^T_{\sigma c_s} \\ \tilde{G}^T_{\sigma c_1} \\ \vdots \\ \tilde{G}^T_{\sigma c_s} \end{bmatrix} \tag{B.45}
$$

which can be rearranged to show Q_{σ_c} as a variable, and then inverted, yielding equation B.44. Now, all that is left is to show that the coefficient matrix in equation B.44 is, in fact, invertible. The proof of invertibility follows closely the proof of the Uncontrollable/Controllable theorem in Chapter V. We begin by noting that row exchanges can be used to put every row which has a non-zero element of the matrix D^T_{qc} in order on the bottom of the matrix. This yields a new matrix which is block lower triangular, where the lower right block matrix is the $ps \times ps$ identity matrix, and where the upper left block is the matrix Δ_{ctrunc} which was shown to be invertible in the proof of the Uncontrollable/Controllable theorem in Chapter V. Therefore, since both diagonal blocks are invertible, the entire new matrix is invertible, and since row exchanges do not alter the invertibility of a matrix, this proves the theorem. ◁

5. UNCONTROLLABLE/MIXED PART

The uncontrollable/mixed portion of the quadratic coordinate transformation process is given by the solutions $H_{\sigma m}$ and $\check{Q}_{\sigma m}$ of equation B.5, subject to the constraints of equation B.7, which are rewritten here for convenience,

$$H_{\sigma m} D_m - F_\sigma H_{\sigma m} = \tilde{Q}_{\sigma m} - \check{Q}_{\sigma m} \tag{B.46}$$

$$\tilde{G}_{\sigma u} - H_{\sigma m} D_{B_\sigma} = 0 \tag{B.47}$$

The quadratic normal form of this portion was proven to include all of the σy_1 terms and nothing else in Chapter V. If we define the σy_1 terms as $Q_{\sigma_m}[\sigma y_1]$ then the matrix $\check{Q}_{\sigma m}$ has the form

$$\check{Q}_{\sigma m} \sigma y^{(2)} = Q_{\sigma_m}[\sigma y_1] \tag{B.48}$$

where the matrix $Q_{\sigma_m} \in R^{s \times s}$ has the block form

$$Q_{\sigma_m} = \begin{bmatrix} 0 \\ Q_{z_m} \\ Q_{w_m} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ Q_{z_{m_1}} & Q_{z_{m_2}} \\ Q_{w_{m_1}} & Q_{w_{m_2}} \end{bmatrix} \tag{B.49}$$

In order to perform the matrix operations we need to define a new matrix which relates $\check{Q}_{\sigma m}$ to Q_{σ_m}. We do this in a lemma.

Lemma 5.1 (Definition of D_{q_m}) *The matrix $D_{q_m} \in R^{s \times ps}$ relates the rows of the matrix $\check{Q}_{\sigma c}$ to the rows of the matrix Q_{σ_c} as follows*

$$\check{Q}_{\sigma c_j} = Q_{\sigma_{c_j}} D_{q_m} \tag{B.50}$$

where D_{q_m} is defined by the relation

$$D_{q_m} = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} \tag{B.51}$$

Proof. By the "Quadratic State Vector" lemma in Chapter V, the elements of the quadratic state vector $\sigma y^{(2)}$ are ordered by the rule $\sigma_h y_i > \sigma_j y_k$ if $i > k$, or

$h > j$ if $i = k$, which gives

$$\sigma y^{(2)} = \begin{bmatrix} \sigma y_1 \\ \sigma y_2 \\ \vdots \\ \sigma y_p \end{bmatrix} \tag{B.52}$$

The matrix D_{q_m} is the matrix which selects only the σy_1 terms from this vector. $\triangleleft$

So now we can state the form of the matrix $\check{Q}_{\sigma m}$, which is

$$\check{Q}_{\sigma m} = \begin{bmatrix} \check{Q}_{\sigma m_1} \\ \vdots \\ \check{Q}_{\sigma m_s} \end{bmatrix} = \begin{bmatrix} Q_{\sigma m_1} D_{q_m} \\ \vdots \\ Q_{\sigma m_s} D_{q_m} \end{bmatrix} \tag{B.53}$$

Applying the Unstacking Theorem from Chapter V to equations B.46 and B.47, we get

$$\left(\begin{bmatrix} D_m^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_m^T \end{bmatrix} - \begin{bmatrix} F_{\sigma_{11}} I & \cdots & F_{\sigma_{1s}} I \\ \vdots & & \vdots \\ F_{\sigma_{s1}} I & \cdots & F_{\sigma_{ss}} I \end{bmatrix} \right) \begin{bmatrix} H_{\sigma m_1}^T \\ \vdots \\ H_{\sigma m_s}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{\sigma m_1}^T \\ \vdots \\ \tilde{Q}_{\sigma m_s}^T \end{bmatrix} - \begin{bmatrix} \check{Q}_{\sigma m_1}^T \\ \vdots \\ \check{Q}_{\sigma m_s}^T \end{bmatrix} \tag{B.54}$$

$$\begin{bmatrix} D_{B_\sigma}^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_{B_\sigma}^T \end{bmatrix} \begin{bmatrix} H_{\sigma m_1}^T \\ \vdots \\ H_{\sigma m_s}^T \end{bmatrix} = \begin{bmatrix} \tilde{G}_{\sigma m_1}^T \\ \vdots \\ \tilde{G}_{\sigma m_s}^T \end{bmatrix} \tag{B.55}$$

The solution to these equations is the subject of the next theorem.

Theorem 5.2 (Uncontrollable/Mixed Solution) *The uncontrollable/mixed portion of the quadratic coordinate transformation required to put a system into quadratic normal form is given by the solutions $H_{\sigma m}$ and $Q_{\sigma m}$ of the block matrix equation*

$$
\begin{bmatrix} H^T_{ym_1} \\ \vdots \\ H^T_{ym_p} \\ Q_{\sigma m_1} \\ \vdots \\ Q_{\sigma m_s} \end{bmatrix} = \begin{bmatrix}
\left(D^T_m - F_{\sigma_{11}} I\right) & -F_{\sigma_{12}} I & \cdots & -F_{\sigma_{1s}} I & D^T_{qm} & 0 & \cdots & 0 \\
-F_{\sigma_{21}} I & \ddots & & \vdots & 0 & \ddots & & \vdots \\
\vdots & & \ddots & -F_{\sigma_{(s-1)s}} I & \vdots & & \ddots & 0 \\
-F_{\sigma_{s1}} I & \cdots & -F_{\sigma_{s(s-1)}} I & \left(D^T_m - F_{\sigma_{ss}} I\right) & 0 & \cdots & 0 & D^T_{qm} \\
D^T_{B_\sigma} & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & \vdots & & & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & & & \vdots \\
0 & \cdots & 0 & D^T_{B_\sigma} & 0 & \cdots & \cdots & 0
\end{bmatrix}^{-1} \begin{bmatrix} \tilde{Q}^T_{\sigma m_1} \\ \vdots \\ \tilde{Q}^T_{\sigma m_s} \\ \tilde{G}^T_{\sigma u_1} \\ \vdots \\ \tilde{G}^T_{\sigma u_s} \end{bmatrix} \tag{B.56}
$$

Proof. We begin by plugging equation B.53 into equation B.54, and appending equations B.54 and B.55 together. The resulting block matrix equation is

$$
\begin{bmatrix}
(D^T_m - F_{\sigma_{11}} I) & -F_{\sigma_{12}} I & \cdots & -F_{\sigma_{1s}} I \\
-F_{\sigma_{21}} I & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & -F_{\sigma_{(s-1)s}} I \\
-F_{\sigma_{s1}} I & \cdots & -F_{\sigma_{s(s-1)}} I & (D^T_m - F_{\sigma_{ss}} I) \\
D^T_{B_\sigma} & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & D^T_{B_\sigma}
\end{bmatrix} \begin{bmatrix} H^T_{ym_1} \\ \vdots \\ H^T_{ym_p} \end{bmatrix} = \begin{bmatrix} \tilde{Q}^T_{\sigma m_1} \\ \vdots \\ \tilde{Q}^T_{\sigma m_s} \\ \tilde{G}^T_{\sigma u_1} \\ \vdots \\ \tilde{G}^T_{\sigma u_s} \end{bmatrix} - \begin{bmatrix} D^T_{q_m} Q^T_{\sigma_{m_1}} \\ \vdots \\ D^T_{q_m} Q^T_{\sigma_{m_s}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{B.57}
$$

which can be rearranged to show $Q_{\sigma m}$ as a variable, and then inverted, yielding equation B.56. Now, all that is left is to show that the coefficient matrix in equation B.56 is, in fact, invertible. The proof of invertibility uses block Gaussian elimination following the proof of the Controllable/Mixed theorem, and uses block row exchanges to invoke the "Invertible Matrix" theorem following the proof of the Uncontrollable/Mixed theorem, all in Chapter V. We first use block row exchanges to put the rows containing the matrices $D^T_{B_\sigma}$ directly below the rows containing the matrices $\left(D^T_m - F_{\sigma_{jj}} I\right)$, and then exchange the rows containing a top block of the matrix D^T_m to the bottom of

the matrix in order, where we have used the fact that

$$D_m^T = \begin{bmatrix} F_\sigma^T & 0 & \cdots & \cdots & 0 \\ I & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I & F_\sigma^T \end{bmatrix} \tag{B.58}$$

This results in a matrix with identity matrices on the main diagonal, and only upper triangular zero main diagonal matrices above the main diagonal of the whole matrix. Block Gaussian elimination can then eliminate all sub-matrices below the main diagonal, resulting in an upper triangular matrix, which is invertible. $\triangleleft$

6. UNCONTROLLABLE/UNCONTROLLABLE PART

The uncontrollable/uncontrollable portion of the quadratic coordinate transformation process is given by the solutions $H_{\sigma u}$ and $\check{Q}_{\sigma u}$ of equation B.6, rewritten here for convenience,

$$H_{\sigma u} D_\sigma - F_\sigma H_{\sigma u} = \tilde{Q}_{\sigma u} - \check{Q}_{\sigma u} \tag{B.59}$$

which is not subject to any of the constraint equations. The quadratic normal form of this portion is the Poincare normal form of the uncontrolled dynamics, and is entirely dependent on the structure of the matrix F_σ. Without a detailed knowledge of the specific problem to be solved, we can only say that $\check{Q}_{\sigma u}$ has the general form

$$\check{Q}_{\sigma u} = \begin{bmatrix} \check{Q}_{\sigma u_1} \\ \vdots \\ \check{Q}_{\sigma u_s} \end{bmatrix} \tag{B.60}$$

Equation B.59 is in the proper form to apply the Unstacking Theorem of Chapter V, which yields

$$\left(\begin{bmatrix} D_\sigma^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_\sigma^T \end{bmatrix} - \begin{bmatrix} F_{\sigma_{11}} I & \cdots & F_{\sigma_{1s}} I \\ \vdots & & \vdots \\ F_{\sigma_{s1}} I & \cdots & F_{\sigma_{ss}} I \end{bmatrix} \right) \begin{bmatrix} H_{\sigma u_1}^T \\ \vdots \\ H_{\sigma u_s}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{\sigma u_1}^T \\ \vdots \\ \tilde{Q}_{\sigma u_s}^T \end{bmatrix} - \begin{bmatrix} \check{Q}_{\sigma u_1}^T \\ \vdots \\ \check{Q}_{\sigma u_s}^T \end{bmatrix} \tag{B.61}$$

Further analysis is not fruitful unless the specifics of the problem are known, which will be taken up in Appendix D.

APPENDIX C. STRUCTURAL MATRICES USED IN CALCULATION OF THE CENTER MANIFOLD

1. INTRODUCTION

There are several matrices which occur in the course of developing the dynamics on the center manifold which do not depend strongly on the details of the problem being considered. Instead, they occur as a consequence of the structure of the solution method being used, and are referred to as "structural" matrices here and in the text. These matrices include the matrix valued functions $M_1(\Pi_L)$, $M_2(\Pi_L)$, $M_3(\Omega_L)$, $M_4(\Omega_L)$, $M_5(\Omega_L, \Pi_L)$, $M_6(\Pi_L)$, $M_7(\Omega_L, \Pi_L)$, and $M_8(\Pi_L)$, and also $N_1(\Pi_Q)$, $N_2(\Pi_Q)$, $N_3(\Omega_L, \Omega_Q)$, $N_4(\Omega_L, \Omega_Q)$, and $N_5(\Omega_L, \Omega_Q, \Pi_L, \Pi_Q)$.

Where do these matrices come from? They are a consequence of the fact that, on the center manifold, both y and w are functions of μ and z, i.e

$$w_{cm} = \Omega(\mu, z) \tag{C.1}$$

$$y_{cm} = \Pi(\mu, z) \tag{C.2}$$

which we can expand in a Taylor series using functional order notation and vector/matrix notation as

$$\begin{aligned} \Omega(\mu, z) &= \Omega^{(1)}(\mu, z) + \Omega^{(2)}(\mu, z) + O^{(3+)} \\ &= \Omega_L \begin{bmatrix} \mu \\ z \end{bmatrix} + \Omega_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \end{aligned} \tag{C.3}$$

and

$$\begin{aligned} \Pi(\mu, z) &= \Pi^{(1)}(\mu, z) + \Pi^{(2)}(\mu, z) + O^{(3+)} \\ &= \Pi_L \begin{bmatrix} \mu \\ z \end{bmatrix} + \Pi_Q \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \end{aligned} \tag{C.4}$$

Here we have used the fact that the zeroth order terms are zero, since the center manifold is tangent to the center subspace, that is, on the center manifold, w and y are zero when μ and z are zero.

Now, in the quadratic normal form of our control system, we have several quadratic state vectors containing elements of w and y which we would like to evaluate on the center manifold as functions of μ and z. These include the quadratic state vectors $w^{(2)}$, μy_1, etc. When we evaluate these on the center manifold by plugging in equations C.1 and C.2, we get quadratic combinations of the terms in equations C.3 and C.4, which can be grouped according to term order. This grouping by term order leads directly to the quadratic matrices M and cubic matrices N which we will calculate in this appendix. We illustrate the process with a brief example.

Example. [Structural Matrices for the Center Manifold] Look at a system with a one dimensional center manifold in y only,

$$
\begin{aligned}
y_{1_{cm}} &= \Pi_L \begin{bmatrix} \mu_1 \\ z_1 \end{bmatrix} + \Pi_Q \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \end{bmatrix} + O^{(3+)} \qquad \text{(C.5)} \\
&= \begin{bmatrix} \Pi_{L_1} & \Pi_{L_2} \end{bmatrix} \begin{bmatrix} \mu_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} \Pi_{Q_1} & \Pi_{Q_2} & \Pi_{Q_3} \end{bmatrix} \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \end{bmatrix} + O^{(3+)}
\end{aligned}
$$

where $\mu = [\mu_1] \in R^1$, $z = [z_1] \in R^1$ and $y = [y_1] \in R^1$. Now, suppose we wanted to evaluate the quadratic state vector $y^{(2)} = [y_1^2]$ on the center manifold. Plugging in equations C.2 and C.4, we get

$$
\begin{aligned}
&\left[y_1^2\right]_{cm} \qquad \text{(C.6)} \\
&= \left(\Pi^{(1)}(\mu, z) + \Pi^{(2)}(\mu, z) + O^{(3+)}\right)^2 \\
&= \left(\Pi^{(1)}(\mu, z)\right)^2 + \left(\Pi^{(1)}(\mu, z)\,\Pi^{(2)}(\mu, z) + \Pi^{(2)}(\mu, z)\,\Pi^{(1)}(\mu, z)\right) + O^{(4+)} \\
&= \left(\Pi_{L_1}\mu_1 + \Pi_{L_2} z_1\right)^2 + 2\left(\Pi_{L_1}\mu_1 + \Pi_{L_2} z_1\right)\left(\Pi_{Q_1}\mu_1^2 + \Pi_{Q_2}\mu_1 z_1 + \Pi_{Q_3} z_1^2\right) + O^{(4+)}
\end{aligned}
$$

where we have grouped all the quadratic order terms together, and all the cubic order terms together. Now, any quadratic order term can be expressed as a matrix of coefficients times the appropriate quadratic state vector, and likewise for the cubic terms. So, we can write

$$y_{cm}^{(2)} = M(\Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N(\Pi_L, \Pi_Q) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{C.7}$$

where

$$M(\Pi_L) = \begin{bmatrix} \Pi_{L_1}^2 & 2\Pi_{L_1}\Pi_{L_2} & \Pi_{L_2}^2 \end{bmatrix} \tag{C.8}$$

and

$$N(\Pi_L, \Pi_Q) = \begin{bmatrix} 2\Pi_{L_1}\Pi_{Q_1} & 2(\Pi_{L_1}\Pi_{Q_2} + \Pi_{L_2}\Pi_{Q_1}) & 2(\Pi_{L_1}\Pi_{Q_3} + \Pi_{L_2}\Pi_{Q_2}) & 2\Pi_{L_2}\Pi_{Q_3} \end{bmatrix} \tag{C.9}$$

We can see that the quadratic matrix of coefficients $M(\Pi_L)$ is a matrix valued function of the elements of the linear center manifold coefficient matrix Π_L, and that the cubic matrix of coefficients $N(\Pi_L, \Pi_Q)$ is a matrix valued function of the elements of both the linear and quadratic center manifold coefficient matrices Π_L and Π_Q. We will use this type of development throughout this appendix. ◁

2. DEFINITIONS OF THE VARIOUS STRUCTURAL MATRICES M AND N

The structural matrices $M_1(\Pi_L)$, $M_2(\Pi_L)$, $M_3(\Omega_L)$, $M_4(\Omega_L)$, $M_5(\Omega_L, \Pi_L)$, $M_6(\Pi_L)$, $M_7(\Omega_L, \Pi_L)$, $M_8(\Pi_L)$, and $N_1(\Pi_Q)$, $N_2(\Pi_Q)$, $N_3(\Omega_L, \Omega_Q)$, $N_4(\Omega_L, \Omega_Q)$, $N_5(\Omega_L, \Omega_Q, \Pi_L, \Pi_Q)$ are defined by the relations

$$\begin{bmatrix} \mu y_1 \\ z y_1 \end{bmatrix}_{cm} \equiv M_1(\Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_1(\Pi_L, \Pi_Q) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{C.10}$$

$$\begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix}_{cm} \equiv M_2\left(\Pi_L\right)\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_2\left(\Pi_L, \Pi_Q\right)\begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{C.11}$$

$$\begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix}_{cm} \equiv M_3\left(\Omega_L\right)\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_3\left(\Omega_L, \Omega_Q\right)\begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{C.12}$$

$$\begin{bmatrix} w^{(2)} \end{bmatrix}_{cm} \equiv M_4\left(\Omega_L\right)\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_4\left(\Omega_L, \Omega_Q\right)\begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{C.13}$$

$$\begin{bmatrix} w y_1 \end{bmatrix}_{cm} \equiv M_5\left(\Omega_L, \Pi_L\right)\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_5\left(\Omega_L, \Omega_Q, \Pi_L, \Pi_Q\right)\begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{C.14}$$

$$\begin{bmatrix} \mu y^{(2)} \\ z y^{(2)} \end{bmatrix}_{cm} \equiv M_6\left(\Pi_L\right)\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \tag{C.15}$$

$$\begin{bmatrix} w y^{(2)} \end{bmatrix}_{cm} \equiv M_7\left(\Omega_L, \Pi_L\right)\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \tag{C.16}$$

$$\begin{bmatrix} y^{(2)} \end{bmatrix}_{cm} \equiv M_8\left(\Pi_L\right)\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \tag{C.17}$$

We will look at the individual matrices in subsequent sections.

3. THE MATRICES M_1 AND N_1

The matrices $M_1(\Pi_L)$ and $N_1(\Pi_Q)$ are defined by equation C.10, which we repeat here for convenience

$$\begin{bmatrix} \mu y_1 \\ z y_1 \end{bmatrix}_{cm} = M_1(\Pi_L)\begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_1(\Pi_Q)\begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{C.18}$$

and where $\mu \in R^r$, $z \in R^q$ and $y \in R^p$. We first look at the case for when $r = 1$ and $q = 1$; then we will look at the case when $r = 1$ and $q = 2$.

a. One Dimensional Case

For $\mu \in R^1$, $z \in R^1$ and $y \in R^p$, that is, for a one dimensional co-dimension one bifurcation, the matrices $M_1(\Pi_L)$ and $N_1(\Pi_Q)$ are defined by the relation

$$\begin{bmatrix} \mu_1 y_1 \\ z_1 y_1 \end{bmatrix}_{cm} = M_1(\Pi_L)\begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \end{bmatrix} + N_1(\Pi_Q)\begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1 z_1^2 \\ z_1^3 \end{bmatrix} + O^{(4+)} \tag{C.19}$$

Using our definition of y on the center manifold from equation C.2 for the one dimensional case, we have

$$\begin{aligned} y_{cm} &= \begin{bmatrix} y_{1_{cm}} \\ \vdots \\ y_{p_{cm}} \end{bmatrix} &\text{(C.20)} \\ &= \begin{bmatrix} \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} \\ \vdots & \vdots \\ \Pi_{L_{p\mu_1}} & \Pi_{L_{pz_1}} \end{bmatrix}\begin{bmatrix} \mu_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} \Pi_{Q_{1\mu_1^2}} & \Pi_{Q_{1\mu_1 z_1}} & \Pi_{Q_{1z_1^2}} \\ \vdots & \vdots & \vdots \\ \Pi_{Q_{p\mu_1^2}} & \Pi_{Q_{p\mu_1 z_1}} & \Pi_{Q_{pz_1^2}} \end{bmatrix}\begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \end{bmatrix} + O^{(3+)} \end{aligned}$$

Now, plugging in for $y_{1_{cm}}$ allows us to calculate the matrices $M_1(\Pi_L)$ and $N_1(\Pi_Q)$, which are given by

$$M_1(\Pi_L) = \begin{bmatrix} \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} & 0 \\ 0 & \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} \end{bmatrix} \tag{C.21}$$

$$N_1(\Pi_Q) = \begin{bmatrix} \Pi_{Q_{1\mu_1^2}} & \Pi_{Q_{1\mu_1 z_1}} & \Pi_{Q_{1z_1^2}} & 0 \\ 0 & \Pi_{Q_{1\mu_1^2}} & \Pi_{Q_{1\mu_1 z_1}} & \Pi_{Q_{1z_1^2}} \end{bmatrix} \tag{C.22}$$

where the notation $\Pi_{L_{ij}}$ and $\Pi_{Q_{ij}}$ indicates the appropriate element of the matrices Π_L and Π_Q respectively.

b. Two Dimensional Case

For $\mu \in R^1$, $z \in R^2$ and $y \in R^p$, that is, for a two dimensional co-dimension one bifurcation, the matrices $M_1(\Pi_L)$ and $N_1(\Pi_Q)$ are defined by the relation

$$\begin{bmatrix} \mu_1 y_1 \\ z_1 y_1 \\ z_2 y_1 \end{bmatrix}_{cm} = M_1(\Pi_L) \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ \mu_1 z_1 \\ z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix} + N_1(\Pi_Q) \begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1^2 z_2 \\ \mu_1 z_1^2 \\ \mu_1 z_1 z_2 \\ \mu_1 z_2^2 \\ z_1^3 \\ z_1^2 z_2 \\ z_1 z_2^2 \\ z_2^3 \end{bmatrix} + O^{(4+)} \tag{C.23}$$

Using our definition of y on the center manifold from equation C.2 for the two dimensional case, we have

$$\begin{aligned} y_{cm} &= \begin{bmatrix} y_{1_{cm}} \\ \vdots \\ y_{p_{cm}} \end{bmatrix} \qquad (C.24) \\ &= \begin{bmatrix} \Pi_{L_{11}} & \Pi_{L_{12}} & \Pi_{L_{13}} \\ \vdots & \vdots & \vdots \\ \Pi_{L_{p1}} & \Pi_{L_{p2}} & \Pi_{L_{p3}} \end{bmatrix} \begin{bmatrix} \mu_1 \\ z_1 \\ z_2 \end{bmatrix} \end{aligned}$$

$$
+ \begin{bmatrix} \Pi_{Q_{11}} & \Pi_{Q_{12}} & \Pi_{Q_{13}} & \Pi_{Q_{14}} & \Pi_{Q_{15}} & \Pi_{Q_{16}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Pi_{Q_{p1}} & \Pi_{Q_{p2}} & \Pi_{Q_{p3}} & \Pi_{Q_{p4}} & \Pi_{Q_{p5}} & \Pi_{Q_{p6}} \end{bmatrix} \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ \mu_1 z_2 \\ z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix} + O^{(3+)}
$$

Now, plugging in for $y_{1_{cm}}$ allows us to calculate the matrices $M_1(\Pi_L)$ and $N_1(\Pi_Q)$, which are given by

$$
M_1(\Pi_L) = \begin{bmatrix} \Pi_{L_{11}} & \Pi_{L_{12}} & \Pi_{L_{13}} & 0 & 0 & 0 \\ 0 & \Pi_{L_{11}} & 0 & \Pi_{L_{12}} & \Pi_{L_{13}} & 0 \\ 0 & 0 & \Pi_{L_{11}} & 0 & \Pi_{L_{12}} & \Pi_{L_{13}} \end{bmatrix} \tag{C.25}
$$

$$
N_1(\Pi_Q) = \begin{bmatrix} \Pi_{Q_{11}} & \Pi_{Q_{12}} & \Pi_{Q_{13}} & \Pi_{Q_{14}} & \Pi_{Q_{15}} & \Pi_{Q_{16}} & 0 & 0 & 0 & 0 \\ 0 & \Pi_{Q_{11}} & 0 & \Pi_{Q_{12}} & \Pi_{Q_{13}} & 0 & \Pi_{Q_{14}} & \Pi_{Q_{15}} & \Pi_{Q_{16}} & 0 \\ 0 & 0 & \Pi_{Q_{11}} & 0 & \Pi_{Q_{12}} & \Pi_{Q_{13}} & 0 & \Pi_{Q_{14}} & \Pi_{Q_{15}} & \Pi_{Q_{16}} \end{bmatrix} \tag{C.26}
$$

where the notation $\Pi_{L_{ij}}$ and $\Pi_{Q_{ij}}$ indicates the appropriate element of the matrices Π_L and Π_Q respectively.

4. THE MATRICES M_2 AND N_2

The matrices $M_2(\Pi_L)$ and $N_2(\Pi_L, \Pi_Q)$ are defined by equation C.11, which we repeat here for convenience

$$
\begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix}_{cm} \equiv M_2(\Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_2(\Pi_L, \Pi_Q) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{C.27}
$$

and where $\mu \in R^r$, $z \in R^q$ and $y \in R^p$. We first look at the case for when $r = 1$ and $q = 1$; then we will look at the case when $r = 1$ and $q = 2$.

a. One Dimensional Case

For $\mu \in R^1$, $z \in R^1$ and $y \in R^p$, that is, for a one dimensional co-dimension one bifurcation, the matrices $M_2(\Pi_L)$ and $N_2(\Pi_L, \Pi_Q)$ are defined by the relation

$$\begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix}_{cm} = M_2(\Pi_L) \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \end{bmatrix} + N_2(\Pi_L, \Pi_Q) \begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1 z_1^2 \\ z_1^3 \end{bmatrix} + O^{(4+)} \tag{C.28}$$

Using our definition of y on the center manifold from equation C.2 for the one dimensional case, we have

$$y_{cm} = \begin{bmatrix} y_{1_{cm}} \\ \vdots \\ y_{p_{cm}} \end{bmatrix} \tag{C.29}$$

$$= \begin{bmatrix} \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} \\ \vdots & \vdots \\ \Pi_{L_{p\mu_1}} & \Pi_{L_{pz_1}} \end{bmatrix} \begin{bmatrix} \mu_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} \Pi_{Q_{1\mu_1^2}} & \Pi_{Q_{1\mu_1 z_1}} & \Pi_{Q_{1z_1^2}} \\ \vdots & \vdots & \vdots \\ \Pi_{Q_{p\mu_1^2}} & \Pi_{Q_{p\mu_1 z_1}} & \Pi_{Q_{pz_1^2}} \end{bmatrix} \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \end{bmatrix} + O^{(3+)}$$

Now, plugging in for y_{cm} allows us to calculate the matrices $M_2(\Pi_L)$ and $N_2(\Pi_L, \Pi_Q)$, which are given by

$$M_2(\Pi_L) = \begin{bmatrix} \left(\Pi_{L_{1\mu_1}}\right)^2 & 2\Pi_{L_{1\mu_1}}\Pi_{L_{1z_1}} & \left(\Pi_{L_{1z_1}}\right)^2 \\ \vdots & \vdots & \vdots \\ \left(\Pi_{L_{p\mu_1}}\right)^2 & 2\Pi_{L_{p\mu_1}}\Pi_{L_{pz_1}} & \left(\Pi_{L_{pz_1}}\right)^2 \end{bmatrix} \tag{C.30}$$

$$N_2(\Pi_L, \Pi_Q) = \begin{bmatrix} n_{1_1} & n_{1_2} & n_{1_3} & n_{1_4} \\ \vdots & \vdots & \vdots & \vdots \\ n_{p_1} & n_{p_2} & n_{p_3} & n_{p_4} \end{bmatrix} \tag{C.31}$$

where the elements of the matrix $N_2(\Pi_L, \Pi_Q)$ are given by the formulas (for $i = 1$ to p)

$$n_{i_1} = 2\Pi_{L_{i\mu_1}}\Pi_{Q_{i\mu_1^2}} \tag{C.32}$$

$$n_{i_2} = 2\left(\Pi_{L_{i\mu_1}}\Pi_{Q_{i\mu_1 z_1}} + \Pi_{L_{iz_1}}\Pi_{Q_{i\mu_1^2}}\right) \tag{C.33}$$

$$n_{i_3} = 2\left(\Pi_{L_{i\mu_1}}\Pi_{Q_{iz_1^2}} + \Pi_{L_{iz_1}}\Pi_{Q_{i\mu_1 z_1}}\right) \tag{C.34}$$

$$n_{i_4} = 2\Pi_{L_{iz_1}}\Pi_{Q_{iz_1^2}} \tag{C.35}$$

and where the notation $\Pi_{L_{ij}}$ and $\Pi_{Q_{ij}}$ indicates the appropriate element of the matrices Π_L and Π_Q respectively. Now, because of the unique structure of one dimensional bifurcations, such that $F_z = 0$ for the general case, we can simplify the matrices $M_2(\Pi_L)$ and $N_2(\Pi_L, \Pi_Q)$ for the one dimensional case. From the Linear Center Manifold Solution theorem of Chapter VI, we have an expression for the rows of the matrix Π_L, which is

$$\Pi_{L_i} = \Pi_{L_1}\begin{bmatrix} 0 & 0 \\ F_\mu & F_z \end{bmatrix}^{i-1} \tag{C.36}$$

For a one-dimensional bifurcation, we plug in $F_z = 0$ and multiply out to get

$$\Pi_{L_1} = \begin{bmatrix} \Pi_{L_{1\mu_1}} & \Pi_{L_{1z_1}} \end{bmatrix} \tag{C.37}$$

$$\Pi_{L_2} = \begin{bmatrix} F_\mu \Pi_{L_{1z_1}} & 0 \end{bmatrix} \tag{C.38}$$

$$\Pi_{L_j} = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{C.39}$$

for $j = 2$ to p (for $p > 2$). Now we can plug the components of Π_L into equations C.30 and C.31 and get our final result for $M_2(\Pi_L)$ and $N_2(\Pi_L, \Pi_Q)$, which is

$$M_2(\Pi_L) = \begin{bmatrix} \left(\Pi_{L_{1\mu_1}}\right)^2 & 2\Pi_{L_{1\mu_1}}\Pi_{L_{1z_1}} & \left(\Pi_{L_{1z_1}}\right)^2 \\ \left(F_\mu \Pi_{L_{1z_1}}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \tag{C.40}$$

$$N_2\left(\Pi_L, \Pi_Q\right) = \begin{bmatrix} n_{1_1} & n_{1_2} & n_{1_3} & n_{1_4} \\ n_{2_1} & n_{2_2} & n_{2_3} & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{C.41}$$

where the elements of the matrix $N_2\left(\Pi_L, \Pi_Q\right)$ are given by the formulas (for $i = 1$ to p)

$$n_{1_1} = 2\Pi_{L_{1\mu_1}}\Pi_{Q_{1\mu_1^2}} \tag{C.42}$$

$$n_{1_2} = 2\left(\Pi_{L_{1\mu_1}}\Pi_{Q_{1\mu_1 z_1}} + \Pi_{L_{1z_1}}\Pi_{Q_{1\mu_1^2}}\right) \tag{C.43}$$

$$n_{1_3} = 2\left(\Pi_{L_{1\mu_1}}\Pi_{Q_{1z_1^2}} + \Pi_{L_{1z_1}}\Pi_{Q_{1\mu_1 z_1}}\right) \tag{C.44}$$

$$n_{1_4} = 2\Pi_{L_{1z_1}}\Pi_{Q_{1z_1^2}} \tag{C.45}$$

$$n_{2_1} = 2F_\mu\Pi_{L_{1z_1}}\Pi_{Q_{2\mu_1^2}} \tag{C.46}$$

$$n_{2_2} = 2F_\mu\Pi_{L_{1z_1}}\Pi_{Q_{2\mu_1 z_1}} \tag{C.47}$$

$$n_{2_3} = 2F_\mu\Pi_{L_{1z_1}}\Pi_{Q_{2z_1^2}} \tag{C.48}$$

b. Two Dimensional Case

For $\mu \in R^1$, $z \in R^2$ and $y \in R^p$, that is, for a two dimensional co-dimension one bifurcation, the matrices $M_2(\Pi_L)$ and $N_2(\Pi_L, \Pi_Q)$ are defined by the relation

$$\begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_p^2 \end{bmatrix}_{cm} = M_2(\Pi_L) \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ \mu_1 z_1 \\ z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix} + N_2(\Pi_L, \Pi_Q) \begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1^2 z_2 \\ \mu_1 z_1^2 \\ \mu_1 z_1 z_2 \\ \mu_1 z_2^2 \\ z_1^3 \\ z_1^2 z_2 \\ z_1 z_2^2 \\ z_2^3 \end{bmatrix} + O^{(4+)} \tag{C.49}$$

Using our definition of y on the center manifold from equation C.2 for the two dimensional case, we have

$$\begin{aligned} y_{cm} &= \begin{bmatrix} y_{1_{cm}} \\ \vdots \\ y_{p_{cm}} \end{bmatrix} \qquad\qquad (C.50) \\ &= \begin{bmatrix} \Pi_{L_{11}} & \Pi_{L_{12}} & \Pi_{L_{13}} \\ \vdots & \vdots & \vdots \\ \Pi_{L_{p1}} & \Pi_{L_{p2}} & \Pi_{L_{p3}} \end{bmatrix} \begin{bmatrix} \mu_1 \\ z_1 \\ z_2 \end{bmatrix} \\ &+ \begin{bmatrix} \Pi_{Q_{11}} & \Pi_{Q_{12}} & \Pi_{Q_{13}} & \Pi_{Q_{14}} & \Pi_{Q_{15}} & \Pi_{Q_{16}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Pi_{Q_{p1}} & \Pi_{Q_{p2}} & \Pi_{Q_{p3}} & \Pi_{Q_{p4}} & \Pi_{Q_{p5}} & \Pi_{Q_{p6}} \end{bmatrix} \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ \mu_1 z_2 \\ z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix} + O^{(3+)} \end{aligned}$$

Now, plugging in for y_{cm} allows us to calculate the matrices $M_2(\Pi_L)$ and $N_2(\Pi_L, \Pi_Q)$, which are given by

$$M_2(\Pi_L) = \begin{bmatrix} (\Pi_{L_{11}})^2 & 2\Pi_{L_{11}}\Pi_{L_{12}} & 2\Pi_{L_{11}}\Pi_{L_{13}} & (\Pi_{L_{12}})^2 & 2\Pi_{L_{12}}\Pi_{L_{13}} & (\Pi_{L_{13}})^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (\Pi_{L_{p1}})^2 & 2\Pi_{L_{p1}}\Pi_{L_{p2}} & 2\Pi_{L_{p1}}\Pi_{L_{p3}} & (\Pi_{L_{p2}})^2 & 2\Pi_{L_{p2}}\Pi_{L_{p3}} & (\Pi_{L_{p3}})^2 \end{bmatrix} \tag{C.51}$$

$$N_2(\Pi_L, \Pi_Q) = \begin{bmatrix} n_{1_1} & n_{1_2} & n_{1_3} & n_{1_4} & n_{1_5} & n_{1_6} & n_{1_7} & n_{1_8} & n_{1_9} & n_{1_{10}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n_{p_1} & n_{p_2} & n_{p_3} & n_{p_4} & n_{p_5} & n_{p_6} & n_{p_7} & n_{p_8} & n_{p_9} & n_{p_{10}} \end{bmatrix} \tag{C.52}$$

where the elements of the matrix $N_2(\Pi_L, \Pi_Q)$ are given by the formulas (for $i = 1$ to p)

$$n_{i_1} = 2\Pi_{L_{i1}}\Pi_{Q_{i1}} \tag{C.53}$$

$$n_{i_2} = 2(\Pi_{L_{i1}}\Pi_{Q_{i2}} + \Pi_{L_{i2}}\Pi_{Q_{i1}}) \tag{C.54}$$

$$n_{i_3} = 2(\Pi_{L_{i1}}\Pi_{Q_{i3}} + \Pi_{L_{i3}}\Pi_{Q_{i1}}) \tag{C.55}$$

$$n_{i_4} = 2(\Pi_{L_{i1}}\Pi_{Q_{i4}} + \Pi_{L_{i2}}\Pi_{Q_{i2}}) \tag{C.56}$$

$$n_{i_5} = 2(\Pi_{L_{i1}}\Pi_{Q_{i5}} + \Pi_{L_{i2}}\Pi_{Q_{i3}} + \Pi_{L_{i2}}\Pi_{Q_{i3}}) \tag{C.57}$$

$$n_{i_6} = 2(\Pi_{L_{i1}}\Pi_{Q_{i6}} + \Pi_{L_{i3}}\Pi_{Q_{i3}}) \tag{C.58}$$

$$n_{i_7} = 2\Pi_{L_{i2}}\Pi_{Q_{i4}} \tag{C.59}$$

$$n_{i_8} = 2(\Pi_{L_{i2}}\Pi_{Q_{i5}} + \Pi_{L_{i3}}\Pi_{Q_{i4}}) \tag{C.60}$$

$$n_{i_9} = 2(\Pi_{L_{i2}}\Pi_{Q_{i6}} + \Pi_{L_{i3}}\Pi_{Q_{i5}}) \tag{C.61}$$

$$n_{i_{10}} = 2\Pi_{L_{i3}}\Pi_{Q_{i6}} \tag{C.62}$$

and where the notation $\Pi_{L_{ij}}$ and $\Pi_{Q_{ij}}$ indicates the appropriate element of the matrices Π_L and Π_Q respectively.

5. THE MATRICES M_3, M_4, M_5, M_7 AND N_4

The matrices $M_3(\Omega_L)$, $M_4(\Omega_L)$, $M_5(\Omega_L, \Pi_L)$, $M_7(\Omega_L, \Pi_L)$ and $N_4(\Omega_L, \Omega_Q)$ are defined by equations C.12, C.13, C.14 and C.16 which we repeat here for conve-

nience

$$\begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix}_{cm} \equiv M_3(\Omega_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_3(\Omega_L, \Omega_Q) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{C.63}$$

$$\begin{bmatrix} w^{(2)} \end{bmatrix}_{cm} \equiv M_4(\Omega_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_4(\Omega_L, \Omega_Q) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{C.64}$$

$$\begin{bmatrix} w y_1 \end{bmatrix}_{cm} \equiv M_5(\Omega_L, \Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + N_5(\Omega_L, \Omega_Q, \Pi_L, \Pi_Q) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{C.65}$$

$$\begin{bmatrix} w y^{(2)} \end{bmatrix}_{cm} \equiv M_7(\Omega_L, \Pi_L) \begin{bmatrix} \mu^{(2)} \\ \mu z^{(2)} \\ z^{(2)} \end{bmatrix} + O^{(3+)} \tag{C.66}$$

Now, since each element of the matrices $M_3(\Omega_L)$, $M_4(\Omega_L)$, $M_5(\Omega_L, \Pi_L)$, $M_7(\Omega_L, \Pi_L)$ and $N_4(\Omega_L, \Omega_Q)$ contains an element of Ω_L as a factor, and since it was shown in Chapter VI that $\Omega_L = 0$ always, then each of these matrices is zero. That is,

$$M_3(\Omega_L) = 0 \in R^{m(r+q)\times\frac{(r+q)(r+q+1)}{2}} \tag{C.67}$$

$$M_4(\Omega_L) = 0 \in R^{\frac{m(m+1)}{2}\times\frac{(r+q)(r+q+1)}{2}} \tag{C.68}$$

$$M_5(\Omega_L, \Pi_L) = 0 \in R^{m\times\frac{(r+q)(r+q+1)}{2}} \tag{C.69}$$

$$M_7(\Omega_L, \Pi_L) = 0 \in R^{mp\times\frac{(r+q)(r+q+1)}{2}} \tag{C.70}$$

$$N_4(\Omega_L, \Omega_Q) = 0 \in R^{\frac{m(m+1)}{2}\times\frac{(r+q)(r+q+1)(r+q+2)}{6}} \tag{C.71}$$

6. THE MATRICES M_6, M_8, N_3, N_5 AND $\tilde{C}_Z$

The matrices $M_6(\Pi_L)$ and $M_8(\Pi_L)$ are defined by equations C.15 and C.17 and were shown in the corollary to the Center Manifold Theorem in Chapter VI to have no effect in our general method of bifurcation control if the quadratic gain vectors

$K_{\mu y^{(2)}}$, $K_{zy^{(2)}}$ and $K_{y^{(2)}}$ were set to zero, which can be done in all cases without loss of generality. We will not calculate $M_6(\Pi_L)$ or $M_8(\Pi_L)$ in this appendix. If calculation of these matrices should become necessary, then the method used to calculate the matrices $M_1(\Pi_L)$ and $M_2(\Pi_L)$ can be used with success.

The matrices $N_3(\Omega_L, \Omega_Q)$ and $N_5(\Omega_L, \Omega_Q, \Pi_L, \Pi_Q)$ are defined by equations C.12 and C.14, which we repeat here for convenience, where we have included the fact that both $M_3(\Omega_L)$ and $M_5(\Omega_L, \Pi_L)$ are zero as shown in the previous section

$$\begin{bmatrix} \mu w^{(2)} \\ z w^{(2)} \end{bmatrix}_{cm} \equiv N_3(\Omega_L, \Omega_Q) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{C.72}$$

$$\begin{bmatrix} w y_1 \end{bmatrix}_{cm} \equiv N_5(\Omega_L, \Omega_Q, \Pi_L, \Pi_Q) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} + O^{(4+)} \tag{C.73}$$

and where $\mu \in R^r$, $z \in R^q$, $w \in R^m$ and $y \in R^p$. Now, the situation for both matrices is simplified because of the fact that $\Omega_L = 0$, and we get

$$N_3(\Omega_L, \Omega_Q) = N_3(\Omega_Q) \tag{C.74}$$

$$N_5(\Omega_L, \Omega_Q, \Pi_L, \Pi_Q) = N_5(\Pi_L, \Omega_Q) \tag{C.75}$$

since only cubic order terms which don't include Ω_L survive.

The matrices $N_3(\Omega_Q)$ and $N_5(\Pi_L, \Omega_Q)$ are used in the calculation of the matrix $\tilde{C}_z(\Pi_L)$, which is defined by the relation

$$\begin{aligned} \tilde{C}_z(\Pi_L) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} &= \left(Q_{zP_2} N_3(\Omega_Q) + Q_{zm_2} N_5(\Pi_L, \Omega_Q)\right) \begin{bmatrix} \mu^{(3)} \\ \mu z^{(3)} \\ z^{(3)} \end{bmatrix} \\ &\quad + f_z^{(3)}\left(\mu, z, w_{cm}^{(1)}, y_{cm}^{(1)}\right) + g_z^{(2)}\left(\mu, z, w_{cm}^{(1)}, y_{cm}^{(1)}\right) v^{(1)} \end{aligned} \tag{C.76}$$

Although it is possible to calculate the matrices $N_3(\Omega_Q)$ and $N_5(\Pi_L, \Omega_Q)$ in the general case, ultimately it is the matrix $\tilde{C}_z(\Pi_L)$ we care about, and we calculate that

on a case-by-case basis for the specific system we are analyzing. So, the matrices $N_3(\Omega_Q)$ and $N_5(\Pi_L, \Omega_Q)$ should be calculated as a part of the calculation of the matrix $\tilde{C}_z(\Pi_L)$.

APPENDIX D. POINCARE NORMAL FORMS FOR COMMON BIFURCATIONS

In this appendix we will present the Poincare normal forms and associated formulas for various common types of bifurcations. For each type of bifurcation, the Poincare normal form will be given, then the coordinate transformation which achieves the normal form, then the formula which is used to achieve the coefficients of the normal form from the non-transformed system. This appendix will cover saddle-node, transcritical, pitchfork and Hopf bifurcations.

1. SADDLE-NODE BIFURCATIONS

A dynamic system of the form

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ F_\mu & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ z_1 \end{bmatrix} + Q \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \end{bmatrix} + C \begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1 z_1^2 \\ z_1^3 \end{bmatrix} + O^{(4+)} \tag{D.1}$$

with $F_\mu \neq 0$, and where Q and C are coefficient matrices having the form

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ q_1 & q_2 & q_3 \end{bmatrix} \tag{D.2}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix} \tag{D.3}$$

can be transformed into the cubic order Poincare normal form

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{\hat{z}}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ F_\mu & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \hat{z}_1 \end{bmatrix} + Q_P \begin{bmatrix} \mu_1^2 \\ \mu_1 \hat{z}_1 \\ \hat{z}_1^2 \end{bmatrix} + C_P \begin{bmatrix} \mu_1^3 \\ \mu_1^2 \hat{z}_1 \\ \mu_1 \hat{z}_1^2 \\ \hat{z}_1^3 \end{bmatrix} + O^{(4+)} \tag{D.4}$$

The matrices Q_P and C_P are the quadratic and cubic Poincare normal form coefficient matrices respectively, having the form

$$Q_P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & q_3^* \end{bmatrix} \tag{D.5}$$

$$C_P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4^* \end{bmatrix} \tag{D.6}$$

The system experiences a saddle-node bifurcation at $\mu_1 = 0$ when $q_3^* \neq 0$. If $q_3^* = 0$, but $c_4^* \neq 0$, then the system does not experience a bifurcation, but is characterized by a single equilibrium point whose stability is dependent on the sign of c_4^*, as discussed in Chapter VII. The quadratic coordinate transformation

$$\begin{bmatrix} \mu_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \hat{z}_1 \end{bmatrix} + H \begin{bmatrix} \mu_1^2 \\ \mu_1 \hat{z}_1 \\ \hat{z}_1^2 \end{bmatrix} \tag{D.7}$$

suffices to put the quadratic terms of the system into normal form, where H is a coefficient matrix of the form

$$H = \begin{bmatrix} 0 & 0 & 0 \\ h_1 & h_2 & h_3 \end{bmatrix} \tag{D.8}$$

with the coefficients h_i given by the formulas

$$h_1 = 0 \tag{D.9}$$

$$h_2 = \frac{q_1}{F_\mu} \tag{D.10}$$

$$h_3 = \frac{q_2}{2F_\mu} \tag{D.11}$$

Finally, the coefficients of the Poincare normal form are given by the formulas

$$q_3^* = q_3 \tag{D.12}$$

$$c_4^* = c_4 \tag{D.13}$$

2. ONE DIMENSIONAL DEGENERATE BIFURCATIONS

The one dimensional degenerate bifurcations include transcritical bifurcations, pitchfork bifurcations, and the case of the isolated equlibrium point. Each of these cases is characterized by a dynamic system of the form

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{z}_1 \end{bmatrix} = Q \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ z_1^2 \end{bmatrix} + C \begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1 z_1^2 \\ z_1^3 \end{bmatrix} + O^{(4+)} \tag{D.14}$$

where Q and C are coefficient matrices having the form

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ q_1 & q_2 & q_3 \end{bmatrix} \tag{D.15}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix} \tag{D.16}$$

The dynamic system in equation D.14 exhibits different dynamic behavior depending on the relationship among the quadratic coefficients q_1, q_2 and q_3. We list some of the possible cases as:

- For $q_3 \neq 0$ and $(q_2)^2 - 4q_3q_1 > 0$, two distinct local equilibrium points exist (except at $\mu_1 = 0$, where there is only one). This situation characterizes a transcritical bifurcation.
- For $(q_2)^2 - 4q_3q_1 < 0$, only one equilibrium point exists, at $\mu_1 = 0$, and no equilibrium points exist for $\mu_1 \neq 0$. This situation characterizes the case of an isolated equilibrium point.
- For $q_3 = 0$, a single local equilibrium point exists at $z^* = -\frac{q_1}{q_2}\mu_1$ (for $q_2 \neq 0$), except when $\mu_1 = 0$, in which case $z^* = arbitrary$. This situation characterizes a pitchfork bifurcation.

Other more degenerate cases are also possible. However, in each case, the dynamic system in equation D.14 can be transformed into the cubic order Poincare

normal form

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{\hat{z}}_1 \end{bmatrix} = Q \begin{bmatrix} \mu_1^2 \\ \mu_1 \hat{z}_1 \\ \hat{z}_1^2 \end{bmatrix} + C_P \begin{bmatrix} \mu_1^3 \\ \mu_1^2 \hat{z}_1 \\ \mu_1 \hat{z}_1^2 \\ \hat{z}_1^3 \end{bmatrix} + O^{(4+)} \tag{D.17}$$

where C_P is the cubic Poincare normal form coefficient matrix. (Because there are no linear terms in equation D.14, the quadratic coefficient matrix Q cannot be altered by the coordinate transformation.) For $q_2 \neq 0$, the matrix C_P has the form

$$C_P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & c_2^* & 0 & c_4^* \end{bmatrix} \tag{D.18}$$

although other forms are possible, particularly if the coefficient $q_2 = 0$. (The degenerate case of $q_2 = 0$ will not be treated here.) The quadratic coordinate transformation

$$\begin{bmatrix} \mu_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \hat{z}_1 \end{bmatrix} + H \begin{bmatrix} \mu_1^2 \\ \mu_1 \hat{z}_1 \\ \hat{z}_1^2 \end{bmatrix} \tag{D.19}$$

suffices to put the quadratic terms of the system into normal form, where H is a coefficient matrix of the form

$$H = \begin{bmatrix} 0 & 0 & 0 \\ h_1 & h_2 & h_3 \end{bmatrix} \tag{D.20}$$

with the coefficients h_i given by the formulas

$$h_1 = -\frac{c_1}{q_2} \tag{D.21}$$

$$h_2 = 0 \tag{D.22}$$

$$h_3 = \frac{c_3}{q_2} \tag{D.23}$$

and where it is assumed that $q_2 \neq 0$. Finally, the coefficients of the Poincare normal form are given by the formulas

$$c_2^* = -2\frac{q_3}{q_2}c_1 - 2\frac{q_1}{q_2}c_3 \tag{D.24}$$

$$c_4^* = c_4 \tag{D.25}$$

3. HOPF BIFURCATIONS

A dynamic system of the form

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega_0 \\ 0 & \omega_0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ z_1 \\ z_2 \end{bmatrix} + Q \begin{bmatrix} \mu_1^2 \\ \mu_1 z_1 \\ \mu_1 z_2 \\ z_1^2 \\ z_1 z_2 \\ z_2^2 \end{bmatrix} + C \begin{bmatrix} \mu_1^3 \\ \mu_1^2 z_1 \\ \mu_1^2 z_2 \\ \mu_1 z_1^2 \\ \mu_1 z_1 z_2 \\ \mu_1 z_2^2 \\ z_1^3 \\ z_1^2 z_2 \\ z_1 z_2^2 \\ z_2^3 \end{bmatrix} + O^{(4+)} \tag{D.26}$$

with $\omega_0 \neq 0$, and where Q and C are coefficient matrices having the form

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ q_{1_1} & q_{1_2} & q_{1_3} & q_{1_4} & q_{1_5} & q_{1_6} \\ q_{2_1} & q_{2_2} & q_{2_3} & q_{2_4} & q_{2_5} & q_{2_6} \end{bmatrix} \tag{D.27}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{1_1} & c_{1_2} & c_{1_3} & c_{1_4} & c_{1_5} & c_{1_6} & c_{1_7} & c_{1_8} & c_{1_9} & c_{1_{10}} \\ c_{2_1} & c_{2_2} & c_{2_3} & c_{2_4} & c_{2_5} & c_{2_6} & c_{2_7} & c_{2_8} & c_{2_9} & c_{2_{10}} \end{bmatrix} \tag{D.28}$$

can be transformed into the cubic order Poincare normal form

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{\hat{z}}_1 \\ \dot{\hat{z}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega_0 \\ 0 & \omega_0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} + Q_P \begin{bmatrix} \mu_1^2 \\ \mu_1 \hat{z}_1 \\ \mu_1 \hat{z}_2 \\ \hat{z}_1^2 \\ \hat{z}_1 \hat{z}_2 \\ \hat{z}_2^2 \end{bmatrix} + C_P \begin{bmatrix} \mu_1^3 \\ \mu_1^2 \hat{z}_1 \\ \mu_1^2 \hat{z}_2 \\ \mu_1 \hat{z}_1^2 \\ \mu_1 \hat{z}_1 \hat{z}_2 \\ \mu_1 \hat{z}_2^2 \\ \hat{z}_1^3 \\ \hat{z}_1^2 \hat{z}_2 \\ \hat{z}_1 \hat{z}_2^2 \\ \hat{z}_2^3 \end{bmatrix} + O^{(4+)} \quad (D.29)$$

where Q_P and C_P are the quadratic and cubic Poincare normal form coefficient matrices respectively, having the form

$$Q_P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1^* & -\omega_1^* & 0 & 0 & 0 \\ 0 & \omega_1^* & \alpha_1^* & 0 & 0 & 0 \end{bmatrix} \quad (D.30)$$

$$C_P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2^* & -\omega_2^* & 0 & 0 & 0 & a_0^* & -b_0^* & a_0^* & -b_0^* \\ 0 & \omega_2^* & \alpha_2^* & 0 & 0 & 0 & b_0^* & a_0^* & b_0^* & a_0^* \end{bmatrix} \quad (D.31)$$

The system experiences a Hopf bifurcation at $\mu_1 = 0$ when $\alpha_1^* \neq 0$ and $a_0^* \neq 0$. The quadratic coordinate transformation

$$\begin{bmatrix} \mu_1 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} + H \begin{bmatrix} \mu_1^2 \\ \mu_1 \hat{z}_1 \\ \mu_1 \hat{z}_2 \\ \hat{z}_1^2 \\ \hat{z}_1 \hat{z}_2 \\ \hat{z}_2^2 \end{bmatrix} \quad (D.32)$$

suffices to put the quadratic terms of the system into normal form, where H is a coefficient matrix of the form

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ h_{1_1} & h_{1_2} & h_{1_3} & h_{1_4} & h_{1_5} & h_{1_6} \\ h_{2_1} & h_{2_2} & h_{2_3} & h_{2_4} & h_{2_5} & h_{2_6} \end{bmatrix} \tag{D.33}$$

with the coefficients h_{ij} given by the formulas

$$h_{1_1} = -\frac{1}{\omega_0} q_{2_1} \tag{D.34}$$

$$h_{2_1} = \frac{1}{\omega_0} q_{1_1} \tag{D.35}$$

$$h_{1_2} = 0 \tag{D.36}$$

$$h_{1_3} = 0 \tag{D.37}$$

$$h_{2_2} = \frac{1}{2\omega_0}(q_{1_2} - q_{2_3}) \tag{D.38}$$

$$h_{2_3} = \frac{1}{2\omega_0}(q_{1_3} + q_{2_2}) \tag{D.39}$$

$$h_{1_4} = \frac{1}{3\omega_0}(-q_{1_5} - q_{2_4} - 2q_{2_6}) \tag{D.40}$$

$$h_{1_5} = \frac{1}{3\omega_0}(2q_{1_4} - 2q_{1_6} + q_{2_5}) \tag{D.41}$$

$$h_{1_6} = \frac{1}{3\omega_0}(q_{1_5} - 2q_{2_4} - q_{2_6}) \tag{D.42}$$

$$h_{2_4} = \frac{1}{3\omega_0}(q_{1_4} + 2q_{1_6} - q_{2_5}) \tag{D.43}$$

$$h_{2_5} = \frac{1}{3\omega_0}(-q_{1_5} + 2q_{2_4} - 2q_{2_6}) \tag{D.44}$$

$$h_{2_6} = \frac{1}{3\omega_0}(2q_{1_4} + q_{1_6} + q_{2_5}) \tag{D.45}$$

The coefficients of the Poincare normal form are given by the formulas

$$\alpha_1^* = \frac{1}{2}(q_{1_2} + q_{2_3}) \tag{D.46}$$

$$\omega_1^* = \frac{1}{2}(-q_{1_3} + q_{2_2}) \tag{D.47}$$

$$a_0^* = \frac{1}{8}\left(3c_{1_7} + c_{1_9} + c_{2_8} + 3c_{2_{10}}\right) + \tilde{a}_0 \tag{D.48}$$

with

$$\tilde{a}_0 = \frac{1}{8\omega_0}\left(q_{1_5}\left(q_{1_4} + q_{1_6}\right) - q_{2_5}\left(q_{2_4} + q_{2_6}\right) - 2q_{1_4}q_{2_4} + 2q_{1_6}q_{2_6}\right) \tag{D.49}$$

We will not be concerned with the cubic order coefficients α_2^*, ω_2^* or b_0^*, as they are essentially higher order terms to lower order dynamics. We also note that equations D.48 and D.49 were adapted from Wiggins [Ref. 21].

APPENDIX E. MATLAB SIMULATION PROGRAM

The following MATLAB program was used to produce the simulation results presented in the first example in Chapter VIII.

```
% Given a system which exhibits a saddle-node bifurcation
% in the absence of control (controllable states held to zero),
% find and apply feedback which stabilizes the system as
% close to the origin as possible.
%
% Our system is:
% mu_dot = 0
% z_dot = mu + 4*z^2 + 5*z*y + y^2
% y_dot = u
%
% We will implement the system two ways:
% (1) Linear control applied to stabilize the controllable
%     state y (u = Ky*y).
% (2) Linear and quadratic control applied to stabilize the
%     controllable state y, and to stabilize the uncontrollable
%     state z by transforming the uncontrollable system into
%         z_dot = mu - 3*z^3 + O^(4)
%     The control needed to implement this is:
%         u = Ky*y + Kz*z + Kzz*z^2

% This section contains all the problem information.
%
% Initial conditions:
mu0 = 0.1;
z0 = 0.1;
y0 = 0;
x0 = [mu0; z0; y0];
%
% Control law gains:
Ky = -10; % Pick Ky at will to stabilize y.
Pi_L_z = -1; % This value is fixed by our system.
% Pick either of these next two gains, not both.
%Kz = 0; %Gain for case (1)
Kz = -Ky*Pi_L_z; %Gain for case (2)
```

```
Pi_Q_zz = -1; % This value can be picked.
Gamma_1_zz = 0;
% Pick either of these next two gains, as above.
%Kzz = 0; %Gain for case (1)
Kzz = -Ky*Pi_Q_zz - Gamma_1_zz; %Gain for case (2)

% This section initializes the numeric integration.
% Tf is the run time of the integrator, dt is the time
% step size.
Tf = 20;
dt = 0.01;
time = 0:dt:Tf;
kmax=length(time);
u = zeros(1, kmax);
x = zeros(3, kmax);
x(:, 1) = x0(:,1);

% This section is the heart of the program. This is
% the numeric integrator where the dynamics are calculated
% and where the control laws are applied.

for (i=1:kmax-1)
mu = x(1,i);
z = x(2,i);
y = x(3,i);

% State propagation and control law application
v = Ky*y + Kz*z + Kzz*z^2;
mu_dot = 0;
z_dot = mu + 4*z^2 + 5*z*y + y^2;
y_dot = v;
mu1 = mu + mu_dot*dt;
z1 = z + z_dot*dt;
y1 = y + y_dot*dt;
x(:,i+1) = [mu1; z1; y1];
u(1,i) = v;
%
end;

% This section plots the results.
clg
plot(x(2,:), x(3,:))
```

```
title('Phase plane plot of z vs y');
xlabel('z'), ylabel('y');
grid;
%print;
pause;
figure
plot(time, x(2,:))
%axis([0, Tf, -1.5, 1.5])
title('forward time plot of z');
xlabel('time (sec)'), ylabel('z');
grid;
pause
figure
plot(time, x(3,:))
%axis([0, Tf, -1.5, 1.5])
title('forward time plot of y');
xlabel('time (sec)'), ylabel('y');
grid;
pause
figure
plot(time, u)
%axis([0, Tf, -1.5, 1.5])
title('forward time plot of u');
xlabel('time (sec)'), ylabel('u');
grid;
pause
figure
subplot(4,1,1), plot(time, x(1,:)); grid;
axis([0, Tf, -0.15, 0.15])
title('Forward time plot of mu, z, y, and u')
xlabel(''), ylabel('mu')

subplot(4,1,2), plot(time, x(2,:)); grid;
%axis([0, Tf, 0, 2])
%title('Forward time plot of z')
xlabel(''), ylabel('z')

subplot(4,1,3), plot(time, x(3,:)); grid;
%title('Forward time plot of y')
xlabel(''), ylabel('y')

subplot(4,1,4), plot(time, u); grid;
```

```
%title('Forward time plot of u')
xlabel('time'), ylabel('u')

%print;
```

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